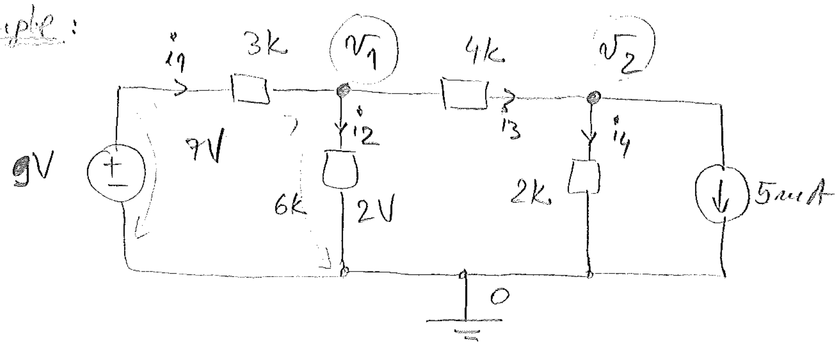
Nodal analysis

- Allows us to find all the node voltages in a circuit.
- Steps: (1) select a reference node and ground it.
  Label nodes whose voltages are unknown.

  (2) Apply KCL at each labeled node, but with
  currents expressed in terms of node voltages via Ohm's law.
  Use voltage polarities according to passive sign convention.

  (3) Solve the resulting set of simultaneous equations to find node voltages.

Example:



$v_1 = ?$
$v_2 = ?$

$$\begin{cases} \dfrac{9-v_1}{3} = \dfrac{v_1}{6} + \dfrac{v_1 - v_2}{4} \\ \dfrac{v_1 - v_2}{4} = \dfrac{v_2}{2} + 5 \end{cases}$$

Resistances kΩ
Currents mA

$$\begin{cases} 36 - 4v_1 = 2v_1 + 3v_1 - 3v_2 \\ v_1 - v_2 = 2v_2 + 20 \end{cases}$$

$$\begin{cases} 9v_1 - 3v_2 = 36 \\ v_1 - 3v_2 = 20 \end{cases} \iff \begin{cases} 3v_1 - v_2 = 12 \\ v_1 - 3v_2 = 20 \end{cases}$$

Written in matrix format:

$$A \cdot x = b \quad , \quad A = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

You can solve this by
- (1) Gaussian elimination (or work with substitutions)
- (2) Use Cramer's rule
- (3) Matlab → not recommended for small systems!

(1) Gaussian elimination:

$$\begin{cases} 3v_1 - v_2 = 12 & /\times\left(-\frac{1}{3}\right) + \\ v_1 - 3v_2 = 20 \end{cases} \Rightarrow \begin{cases} 3v_1 - v_2 = 12 \\ -3v_2 + \frac{v_2}{3} = 20 - 4 = 16 \end{cases} \Rightarrow$$

$$-9v_2 + v_2 = 48 \Leftrightarrow -8v_2 = 48 \Rightarrow \boxed{v_2 = -6}\ [V]$$

$$\Rightarrow 3v_1 + 6 = 12 \Rightarrow \boxed{v_1 = 2}\ [V]$$

(2) Cramer's rule:

$$v_1 = \frac{\begin{vmatrix} 12 & -1 \\ 20 & -3 \end{vmatrix}}{\underbrace{\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}}_{\hat{=}\ \text{determinant of } A}} = \frac{12\times(-3) - 20\times(-1)}{3\times(-3) - 1\times(-1)} = \frac{-36 + 20}{-9 + 1} = \frac{-16}{-8} = 2\ [V]$$

$$v_2 = \frac{\begin{vmatrix} 3 & 12 \\ 1 & 20 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}} = \frac{3\times 20 - 1\times 12}{-8} = \frac{60 - 12}{-8} = \frac{48}{-8} = -6\ [V]$$

$$\boxed{v_1 = 2}\ [V]$$
$$\boxed{v_2 = -6}\ [V]$$

Checking of our results → very important! One way:

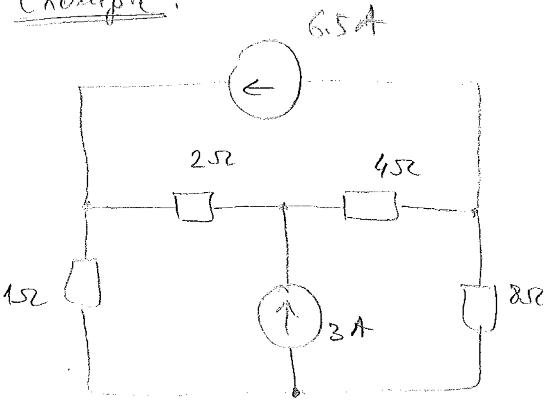
Use calculated node voltages to verify that branch currents satisfy KCL:

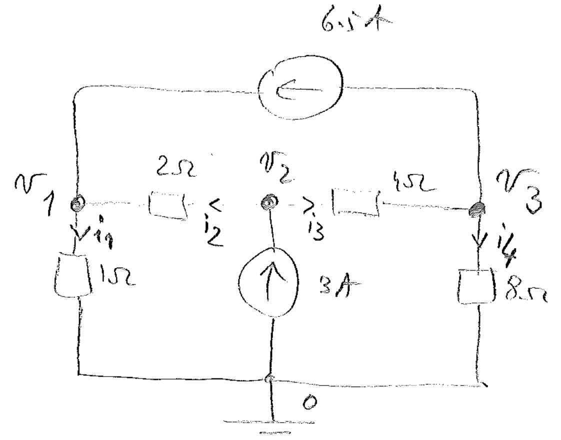
$$i_1 = i_2 + i_3$$

$$\frac{9-2}{3} = \frac{2}{6} + \frac{2-(-6)}{4} \Leftrightarrow \frac{7}{3} = \frac{1}{3} + 2 = \frac{7}{3}\ \checkmark$$

Home: Read Appendix 2: Matrices, Determinants, Cramer's rule!

Example:





$$\begin{cases} 6.5 + i_2 = i_1 \\ 3 = i_2 + i_3 \\ i_3 = i_4 + 6.5 \end{cases} \Leftrightarrow \begin{cases} 6.5 + \dfrac{v_2 - v_1}{2} = \dfrac{v_1}{1} \\ 3 = \dfrac{v_2 - v_1}{2} + \dfrac{v_2 - v_3}{4} \\ \dfrac{v_2 - v_3}{4} = \dfrac{v_3}{8} + 6.5 \end{cases}$$

$$\Rightarrow \begin{cases} 13 + v_2 - v_1 = 2v_1 \\ 12 = 2v_2 - 2v_1 + v_2 - v_3 \\ 2v_2 - 2v_3 = v_3 + 52 \end{cases} \Leftrightarrow \begin{cases} \boxed{3v_1 - v_2 = 13} & (1) \\ -2v_1 + 3v_2 - v_3 = 12 & (2) \\ 2v_2 - 3v_3 = 52 & (3) \end{cases} \Rightarrow$$

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 3 & -1 \\ 0 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \\ 52 \end{bmatrix}$$

$$\Rightarrow v_3 = 3v_2 - 2v_1 - 12 \overset{(3)}{\Rightarrow} 2v_2 - 3(3v_2 - 2v_1 - 12) = 52$$

$$2v_2 - 9v_2 + 6v_1 + 36 = 52$$

$$\boxed{6v_1 - 7v_2 = 16} \Rightarrow$$

$$\begin{cases} 3v_1 - v_2 = 13 \Rightarrow v_2 = -13 + 3v_1 \\ 6v_1 - 7v_2 = 16 \end{cases}$$

$$\Rightarrow 6v_1 - 7(-13 + 3v_1) = 16$$

$$6v_1 + 91 - 21v_1 = 16$$

$$15v_1 = 75 \Rightarrow \boxed{v_1 = 5\,V}$$

$$\Rightarrow v_2 = -13 + 15 = \boxed{2\,V = v_2}$$

$$v_3 = 3 \times 2 - 2 \times 5 - 12 = \boxed{-16\,V = v_3}$$