Sinusoids and Phasors

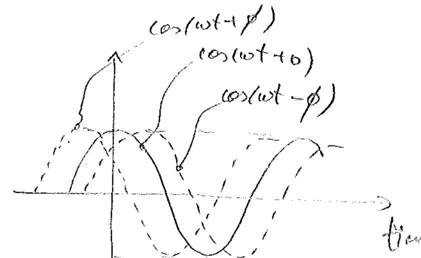
$$x(t) = X_m \sin \frac{2\pi}{T} \cdot t = X_m \sin 2\pi f \cdot t = X_m \sin \omega t$$

period $T = \frac{1}{f}$; frequency (cyclical, ordinary) ; amplitude ; angular frequency

1) General form:

$$\boxed{x(t) = X_m \cdot \cos(\omega t + \phi)}$$

phase angle (rad or degrees) $\boxed{2\pi \text{ rad} = 360°}$



2) For conversion we know

$$\begin{cases} \cos \omega t = \sin(\omega t + \frac{\pi}{2}); \\ \sin \omega t = \cos(\omega t - \frac{\pi}{2}); \end{cases}$$

$$\begin{cases} \cos(u \pm v) = \cos u \cdot \cos v \mp \sin u \cdot \sin v \\ \sin(u \pm v) = \sin u \cdot \cos v \pm \cos u \cdot \sin v \end{cases}$$

3) Phase difference:

$$\begin{cases} x(t) = X_m \cos(\omega t + \phi_x) \\ y(t) = Y_m \cos(\omega t + \phi_y) \end{cases}$$

$$\phi = \phi_y - \phi_x$$

4) Phasors:

- if we apply a sinusoidal signal to a circuit R, L, C and possibly dependent sources, all resulting voltages and currents will oscillate with the same frequency ω, differing only in amplitude and phase angle!
- The goal of $\boxed{\text{ac analysis}}$ (we look only at steady-state response for a sinusoidal forcing function) is to find these amplitudes and phase angles!
- ω is of no concern → is the same.

=> we need a notation which focuses on ampl. & phase alone!

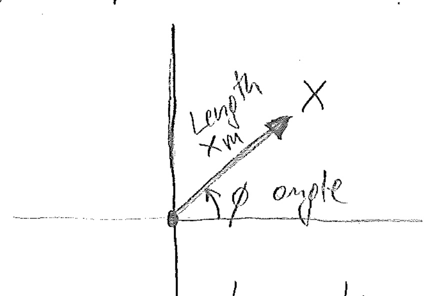
- This is provided by $\boxed{\text{phasor}}$!

time domain representation: $x(t) = X_m \cos(\omega t + \phi)$

$\updownarrow$

freq. domain representation: $\mathbf{X} = X_m \angle\phi$

$$\boxed{\mathbf{X} = X_m \angle\phi} \text{ polar-form}$$



phasor-diagram.

5) Relationship with the $\boxed{\text{complex}}$ form/notation

① $\boxed{\text{HW assignment}}$: write a report on section 10.3!

Complex forcing function:

$$\underbrace{X_m \cos(\omega t+\phi)}_{=Re\left(X_m e^{j(\omega t+\phi)}\right)\text{ Real part}} + \underbrace{jX_m \sin(\omega t+\phi)}_{=Im\left(X_m e^{j(\omega t+\phi)}\right)\text{ imaginary part}} = \boxed{X_m e^{j(\omega t+\phi)}}$$

compact representation → exponential/polar form

(by EULER's formula)

BTW: beautiful $\boxed{e^{j\pi}+1=0}$ ← Euler's number

② $\boxed{\text{HW assignment}}$: Read Appendix 5

Obs: instead of working with only real part, it is convenient to work with complex quantities, which directly relate to the phasor notation!!!
So, instead of applying for example: $v(t)=V_m\cos(\omega t+\phi_v)$
we can apply $v(t)=V_m\cos(\omega t+\phi_v)+jV_m\sin(\omega t+\phi_v)=V_m e^{j(\omega t+\phi_v)}$
and interpret the response also as a complex entity! with real part being the response to the real part of the forcing function!!!

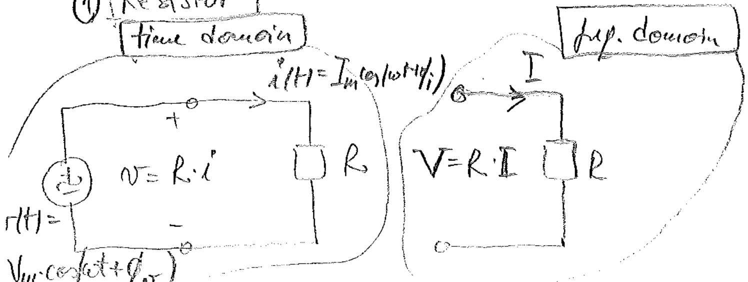
$\boxed{X_m \angle \phi}$ known as the polar form.

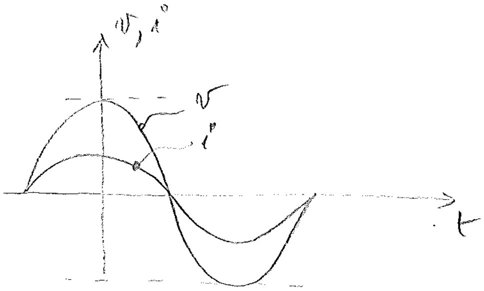
The abbreviated complex representation is the phasor representation.

Phasor relationships for R, L, & C

⚠ The power of the phasor-based analysis is in that it's possible to define algebraic relations between voltage and current for L, C like for R.

① Resistor



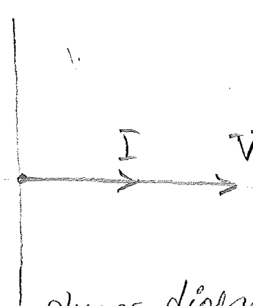


$$v(t)=R\cdot i(t)$$

$$i(t)=\frac{v(t)}{R}=\frac{V_m}{R}\cos(\omega t+\phi_v)$$

$$=I_m\cos(\omega t+\phi_i)$$

$$\boxed{I_m=\frac{V_m}{R}} \quad \boxed{\phi_i=\phi_v}$$



phasor diagram

Obs:
▸ voltage & current are in phase!

To derive the phasor relation, we use complex entities (see verso:)

$v(t) = R \cdot i(t)$

$$v(t) = V_m \cdot e^{j(\omega t + \phi_v)} = R \cdot I_m \cdot e^{j(\omega t + \phi_i)} = R \cdot i(t)$$

$$V_m e^{j\phi_v} = R I_m \cdot e^{j\phi_i}$$

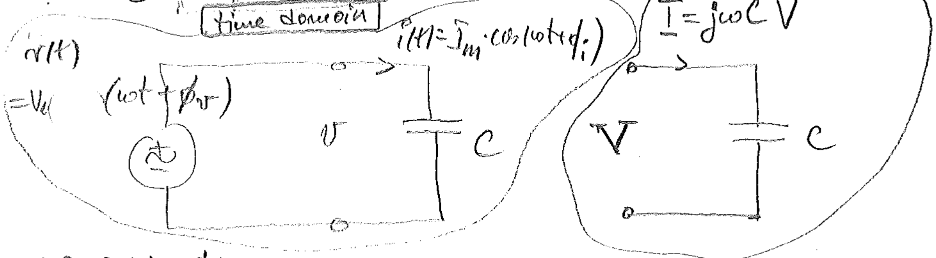
or in polar form:

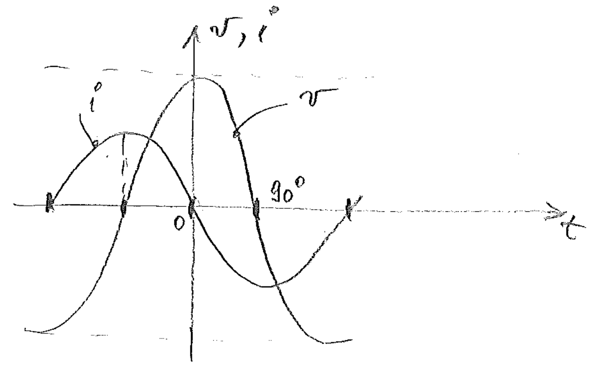
$$V_m \angle \phi_v = R \cdot \boxed{I_m \angle \phi_i}$$

hence, the phasor relation:

$$\boxed{\mathbb{V} = R \cdot \mathbb{I}}$$

② Capacitor





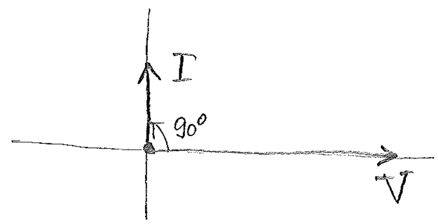
ac current:

$$i(t) = C\frac{dv(t)}{dt} = -\omega C V_m \sin(\omega t+\phi_v)$$

$$= \underbrace{\omega C V_m}_{\uparrow} \cos(\omega t + \underbrace{\phi_v + 90^\circ}_{\uparrow})$$

$$= I_m \cdot \cos(\omega t + \phi_i)$$

$$\boxed{I_m = \omega C V_m} \qquad \boxed{\phi_i = \phi_v + 90^\circ}$$



phasor diagram.

Obs: • amplitudes are still proportional but the constant of proportionality is freq. dependent

• in a capacitance current always leads voltage by 90°

To derive the phasor relation, we start from differential equation:

$$i = C\frac{dv}{dt}$$

and work with complex entities:

$$I_m e^{j(\omega t+\phi_i)} = C\frac{d}{dt}\left[V_m e^{j(\omega t+\phi_v)}\right]$$

$$I_m e^{j(\omega t+\phi_i)} = j\omega C V_m e^{j(\omega t+\phi_v)}$$

$$I_m e^{j\phi_i} = j\omega C V_m e^{j\phi_v}$$

Hence, phasor relation:

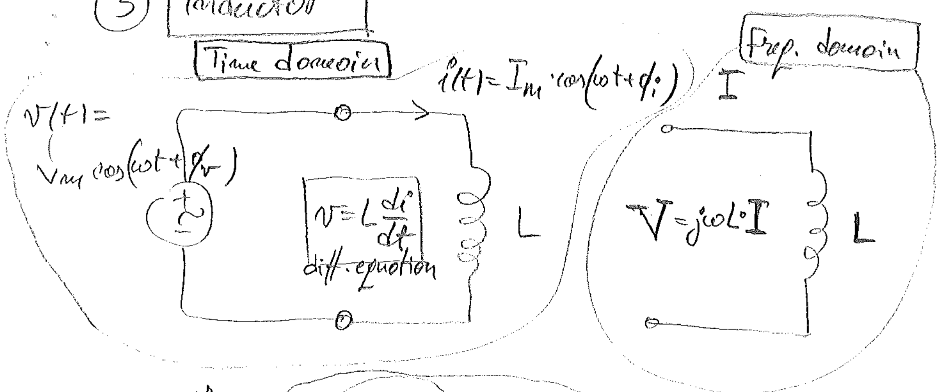
$$\boxed{I = j\omega C\cdot V} \quad \text{or} \quad \boxed{I_m \angle \phi_i = V_m \angle \phi_v + 90^\circ}$$

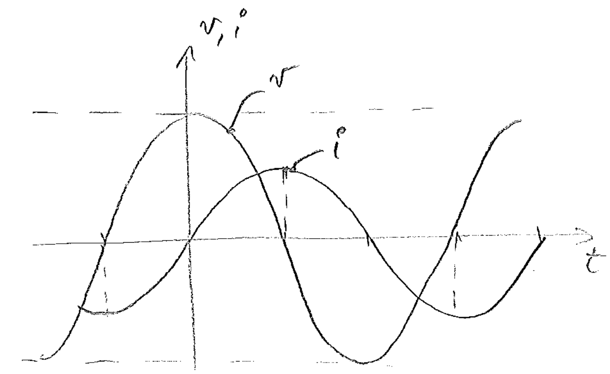
algebraic equation

$\Updownarrow$

$$\boxed{V = \frac{1}{j\omega C} I}$$ mostly used!

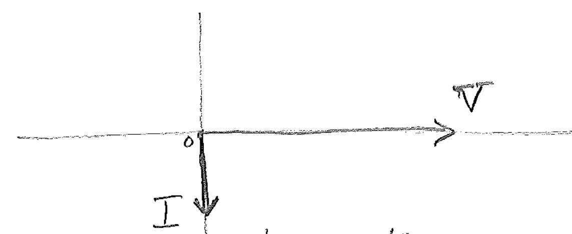
③ Inductor





$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi_i)$$
$$= \omega L I_m \cos(\omega t + \phi_i + 90°)$$
$$= V_m \cdot \cos(\omega t + \phi_v)$$

$$\boxed{I_m = \frac{V_m}{\omega L}} \qquad \boxed{\phi_i = \phi_v - 90°}$$



phasor diagram.

Obs:
- amplitudes proportional; proportionality constant is freq. dependent!
- in an inductance, current always lags voltage by 90°!

To derive phasor relation, we work with complex entities:

$$V_m e^{j(\omega t + \phi_v)} = L \cdot \frac{d}{dt}\left(I_m \cdot e^{j(\omega t + \phi_i)}\right)$$

$$V_m e^{j(\omega t + \phi_v)} = j\omega L I_m \cdot e^{j(\omega t + \phi_i)}$$

$$V_m e^{j\phi_v} = j\omega L I_m e^{j\phi_i}$$

hence, phasor relation: $\boxed{\mathbf{V} = j\omega L \cdot \mathbf{I}}$ or $\boxed{V_m \angle \phi_v = I_m \angle \phi_i + 90°}$

mostly algebraic equation used

$\Updownarrow$

$$\boxed{\mathbf{I} = \frac{1}{j\omega L}\mathbf{V}}$$

④ Limiting cases

Capacitance: $I_m = \omega C V_m \Rightarrow \omega \to 0$: $I_m \to 0$: C is open at low frequencies and short at high freq.

Inductance: $V_m = \omega L I_m \Rightarrow \omega \to 0$: $V_m \to 0$: L is short at low frequencies and open at high freq.

$I_m = \frac{1}{\omega L} V_m$