Response to DC and AC forcing functions

$x(t) \neq 0$

General solution to the Differential Equation

$$\tau \frac{dy(t)}{dt} + y(t) = x(t) \quad \Big/ \times \frac{1}{\tau} e^{\frac{t}{\tau}}$$

$$\underbrace{\frac{dy(t)}{dt} \cdot e^{\frac{t}{\tau}} + \frac{1}{\tau} y(t) \cdot e^{\frac{t}{\tau}}} = \frac{1}{\tau} x(t) \cdot e^{\frac{t}{\tau}}$$

$$\frac{d}{dt}\left[y(t) \cdot e^{\frac{t}{\tau}}\right] = \frac{1}{\tau} x(t) \cdot e^{\frac{t}{\tau}}$$

Replace t with dummy variable ξ and integrate both sides from 0 to t.

$$\int_0^t \frac{d}{d\xi}\left[y(\xi) \cdot e^{\frac{\xi}{\tau}}\right] d\xi = \int_0^t \frac{1}{\tau} x(\xi) \cdot e^{\frac{\xi}{\tau}} d\xi$$

$$y(t) \cdot e^{\frac{t}{\tau}} - y(0) \cdot e^0 = \frac{1}{\tau} \int_0^t x(\xi) \cdot e^{\frac{\xi}{\tau}} d\xi$$

$y(0)$: initial value of $y(t)$

crucial!

$$\boxed{y(t) = \underbrace{y(0) e^{-\frac{t}{\tau}}}_{\text{(I)}} + \underbrace{\frac{1}{\tau} e^{-\frac{t}{\tau}} \int_0^t x(\xi) \cdot e^{\frac{\xi}{\tau}} d\xi}_{\text{(II)}}} \qquad (1)$$

Response contains 2 components!

(I) independent of the forcing function! is the familiar $\boxed{\text{natural response}}$.

(II) depends on the particular forcing function $x(t)$ is called the $\boxed{\text{forced response}}$.

Their sum is called the $\boxed{\text{complete response}}$

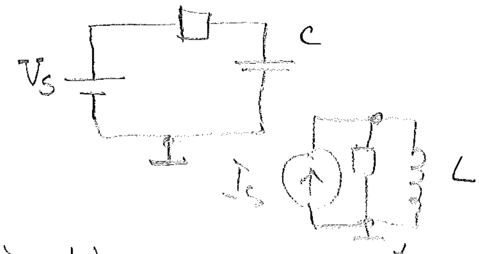
$$\boxed{y(t) = y_{natural} + y_{forced}} \quad (2)$$

$$\boxed{y_{natural} = y(0)\cdot e^{-\frac{t}{\tau}}} \quad (3)$$

$$\boxed{y_{forced} = \frac{1}{\tau} e^{-\frac{t}{\tau}} \int_0^t x(\xi)\cdot e^{\frac{\xi}{\tau}} d\xi} \quad (4)$$

(a) **Response to a DC forcing function**

$\boxed{x(t) = X_S}$ constant.



Example: $\begin{cases} \text{RC circuit we have } v_S = V_S \\ \text{RL circuit we have } i_S = I_S \end{cases}$

$$y_{forced} = \frac{1}{\tau} e^{-\frac{t}{\tau}} \int_0^t X_S \cdot e^{\frac{\xi}{\tau}} d\xi = \frac{X_S}{\cancel{\tau}} e^{-\frac{t}{\tau}} \cdot \cancel{\tau} \underbrace{\left(e^{\frac{\xi}{\tau}} \Big|_0^t \right)}_{=\left(e^{\frac{t}{\tau}} - 1\right)} = X_S\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\boxed{y_{forced} = X_S\left(1 - e^{-\frac{t}{\tau}}\right)} \quad (5)$$

Hence, the complete response:

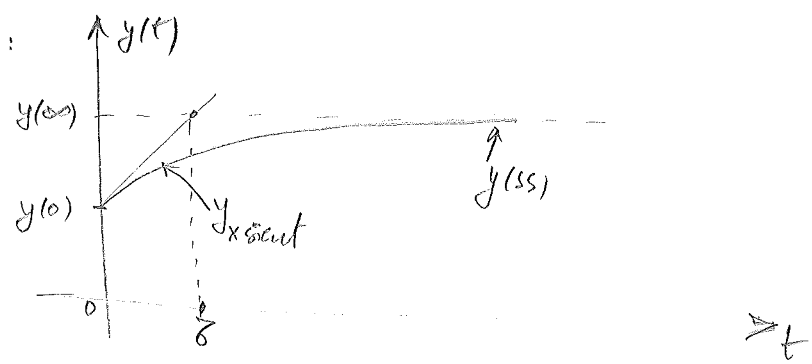
$$\boxed{y(t) = \underbrace{y(0) e^{-\frac{t}{\tau}}}_{= y_{natural}} + \underbrace{X_S\left(1 - e^{-\frac{t}{\tau}}\right)}_{= y_{forced}}} \quad (6)$$

Note that in the limit $t \to \infty$: $y(t) \to \underline{X_S = y(\infty)}$ we can rewrite (6) as:

$$\boxed{y(t) = \left[y(0) - y(\infty) \right] e^{-\frac{t}{\tau}} + y(\infty)} \quad (6')$$

The complete response is an <u>exponential</u> <u>transition</u> from the initial value $y(0)$ to the final value $y(\infty)$! This transition is governed by the <u>time constant</u> τ.

<u>Example</u>: when $y(\infty) > y(0)$:



Seen like this: $$\boxed{y(t) = \underbrace{[y(0) - y(\infty)] \cdot e^{-\frac{t}{\tau}}}_{y_{xsient}} + \underbrace{y(\infty)}_{y_{ss}}} \quad (6')$$

transient component | steady-state component

$$\boxed{y_{xsient} = [y(0) - y(\infty)] \cdot e^{-\frac{t}{\tau}}} \quad (7)$$

$$\boxed{y_{ss} = y(\infty)} \quad (8)$$

Observation: it is common to refer to the complete response to a DC forcing function as simply the transient response!

(b) Response to an AC forcing function

$x(t) = X_m \cdot \cos \omega t$

amplitude — angular frequency of the ac signal. [radians/s] [rad/s]

Example: $\begin{cases} v_s = V_m \cdot \cos \omega t & \text{for RC circuit} \\ i_s = I_m \cdot \cos \omega t & \text{for RL circuit.} \end{cases}$

$$y_{forced} = \frac{X_m}{\tau} \cdot e^{-\frac{t}{\tau}} \cdot \int_0^t e^{\frac{\xi}{\tau}} (\cos \omega \xi)\, d\xi$$

Use formula: $$\boxed{\int e^{a\xi} (\cos b\xi)\, d\xi = \frac{e^{a\xi}}{a^2 + b^2} (a \cos b\xi + b \sin b\xi)}$$

In our case $a = \frac{1}{\tau}$, $b = \omega$

$$y_{forced} = \frac{X_m}{\tau} e^{-\frac{t}{\tau}} \times \frac{e^{\frac{\xi}{\tau}}}{\frac{1}{\tau^2} + \omega^2} \left(\frac{1}{\tau} \cos \omega \xi + \omega \sin \omega \xi \right) \Bigg|_0^t$$

$$\boxed{y_{forced} = \frac{X_m}{1 + (\omega\tau)^2} \left(\cos \omega t + \omega\tau \sin \omega t - e^{-\frac{t}{\tau}} \right)} \quad (9)$$

Use trigonometric identity:

$$A\cos\alpha + B\sin\alpha = \sqrt{A^2+B^2}\cdot\cos\left[\alpha - \tan^{-1}\frac{B}{A}\right]$$

with $A=1$, $B=\omega\tau$

$$\boxed{y_{forced} = -\frac{X_m}{1+(\omega\tau)^2}\cdot e^{-\frac{t}{\tau}} + \frac{X_m}{\sqrt{1+(\omega\tau)^2}}\cos(\omega t - \tan^{-1}\omega t)} \quad (10)$$

Finally the complete response:

$$\boxed{y(t) = \underbrace{\left[y(0) - \frac{X_m}{1+(\omega\tau)^2}\right]\cdot e^{-\frac{t}{\tau}}}_{=y_{xsient}} + \underbrace{\frac{X_m}{\sqrt{1+(\omega\tau)^2}}\cdot\cos(\omega t - \tan^{-1}\omega\tau)}_{=\boxed{y_{ss} = Y_m\cdot\cos(\omega t+\phi)}}} \quad (11)$$

where we introduced notation:

$$\underbrace{Y_m = \frac{X_m}{\sqrt{1+(\omega\tau)^2}}}_{\triangleq \text{ amplitude}} \; ; \quad \underbrace{\phi = -\tan^{-1}\omega\tau}_{\triangleq \text{ phase angle}}.$$

of the steady-state response.

frequency dependent!

So, again: $\boxed{y(t) = y_{xsient} + y_{ss}}$

Observations:

It is common practice to refer to the steady-state response to an ac forcing function as simply the ac response!