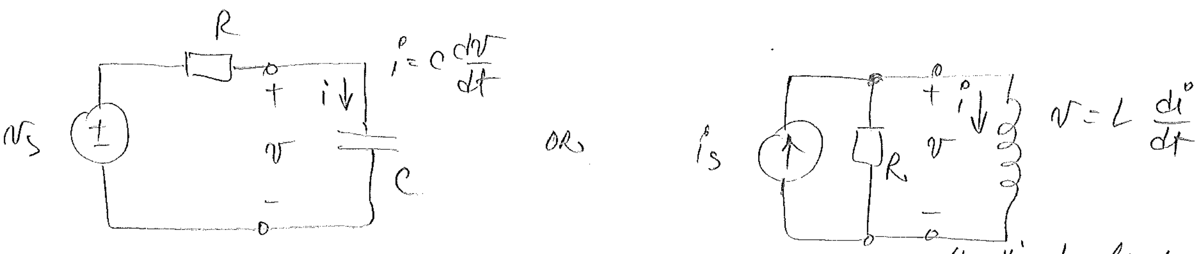
Natural response of RL and RC circuits

- Analysis of RLC circuits still done using KCL & KVL
- However C, L have i-v characteristics that depend on time, equations are not anymore algebraic, they involve time derivatives, or integrals or both!

- When we have only one capacitor or inductor the circuit could be reduced using Thévenin or Norton equivalents to:



(a) "capacitive" case

(b) "inductive" case.

Basic first-order circuits! whose functioning is governed by first-order differential equations

(a)

$$i = C\frac{dv}{dt}$$

KVL: $v_S = R\cdot i + v = RC\frac{dv}{dt} + v$

$$\boxed{RC\frac{dv}{dt} + v = v_S} \quad (1)$$

(b)

$$v = L\frac{di}{dt}$$

KCL: $i_S = \frac{v}{R} + i = \frac{L}{R}\frac{di}{dt} + i$

$$\boxed{\frac{L}{R}\frac{di}{dt} + i = i_S} \quad (2)$$

Note that both equations are of the type:

$$\boxed{\tau \cdot \frac{dy(t)}{dt} + y(t) = x(t)} \quad (3)$$

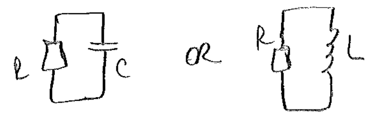
has the dimension of time [s] — the unknown variable — the "forcing" function or the input.

This is a differential equation of the first order. ⇓ first-order circuits!

$$\boxed{\tau = RC} \quad (4) \qquad \boxed{\tau = \frac{L}{R}} \quad (5)$$

We'll solve first (3) for the simple case when $x(t) = 0$!

The $\boxed{\text{source-free}}$ or $\boxed{\text{natural response}}$



- Let: $x(t) = 0$ (cond. this $v_S = 0$ or $i_S = 0$!)

$$(3) \Rightarrow \boxed{\tau \frac{dy(t)}{dt} + y(t) = 0} \quad (6)$$

- This is homogeneous differential equation.
- its solution is the homogeneous solution, or the $\boxed{\text{source-free}}$ solution for the RC and RL circuits!

$$y = -\tau \frac{dy}{dt}$$ (the unknown and its derivative must be the same!)

- Recall from calculus, that only the exponential function enjoys the unique property that its derivative is still exponential!
- So, we assume or guess (see pp. 258 of textbook) a solution of the type:

$$\boxed{y(t) = A \cdot e^{st}} \quad (7)$$

($e = 2.718$ the base of natural logarithms)

Still, we need to find $A, s = ?$

(i) Substitute (7) in (6):

$\boxed{s = ?}$

$$\tau \cdot s \cdot A \cdot e^{st} + A e^{st} = 0$$

$$(\tau s + 1) \cdot A e^{st} = 0$$

- we seek a solution $A e^{st} \neq 0$. Therefore: $\boxed{\tau s + 1 = 0}$ (8) $\Rightarrow \boxed{s = -\frac{1}{\tau}}$ (9)

$\triangleq$ the characteristic equation!

- $\boxed{s = -\frac{1}{\tau}}$ has the dimensions of the reciprocal of time.

$\Rightarrow \boxed{s}$ is called the $\boxed{\text{natural frequency}}$, the $\boxed{\text{characteristic freq}}$ or the $\boxed{\text{critical frequency}}$.

$\left[\frac{Np}{s}\right]$ neper/s

Np is dimensionless.

(ii) Use the initial condition $y(0)$ in the circuit =

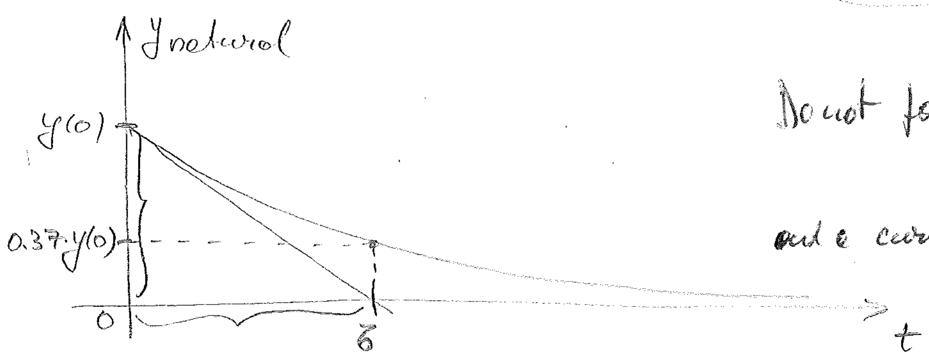
$\boxed{A = ?}$

$= \left\{ \begin{array}{l} \text{initial voltage across the capacitor in the circuit} \\ \text{initial current thru the inductance — '' — ''} \end{array} \right\}$ which are related to the initial stored energy: $\left\{ \begin{array}{l} w(t) = \frac{C v^2(0)}{2} \\ w(t) = \frac{L i^2(0)}{2} \end{array} \right.$

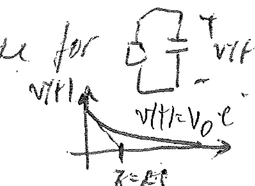
- So, let $t \to 0$ in (7) $\Rightarrow$ $y(0) = A$
- So, finally the solution is $\boxed{y(t) = y(0)\cdot e^{-\frac{t}{\tau}}}$ (10) Extremely important!

$y(0) = V_0$ for circuit (a) RC
$y(0) = I_0$ for circuit (b) RL

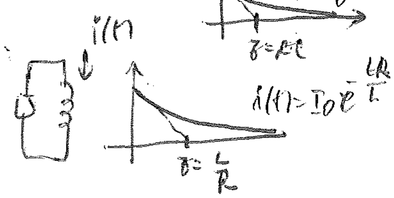
- $y(t)$ is an exponentially decaying function from initial value $y(0)$ to final value $y(\infty) = 0$.
- This decay depends only on $y(0)$ and τ! which are peculiar characteristics of the circuit irrespective of any particular forcing function, this solution is called the $\boxed{\text{natural response}}$.
or $\boxed{\text{source-free response}}$
or $\boxed{\text{homogeneous solution}}$!



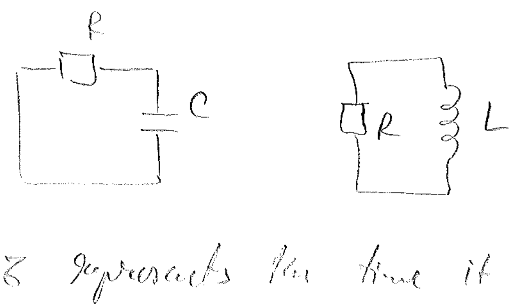
Do not forget y is a voltage for



and a current for:



$\boxed{\tau = RC}$ $\boxed{\tau = \frac{L}{R}}$ is a measure of how rapidly the exponential decay takes place!



(1) τ represents the time it takes for the natural response to decay to $\frac{1}{e} \cong 37\%$ of its initial value
(2) τ represents the time at which the tangent to the natural response in the origin intercepts t axis.