3 Convolution

$f(t) \longleftrightarrow F(s)$

$g(t) \longleftrightarrow G(s)$

- "Convolution" of $f(t)$ and $g(t)$ is defined as:

$$\boxed{f(t) * g(t) \triangleq \int_{0^-}^{t} f(\xi)\cdot g(t-\xi)\, d\xi} \quad (1)$$

- From earlier lectures:

$$\boxed{f(t) * g(t) \longleftrightarrow F(s)\cdot G(s)} \quad (2)$$

- Convolution in the time-domain corresponds to multiplication in the s-domain.

- This offers an alternative technique to find the inverse Laplace transform of $F(s)\cdot G(s)$!
- This approach comes in very handy especially in cases where we work with transfer functions $H(s)$ and the circuit under scrutiny is in its $\boxed{\text{zero-state}}$, when its s-domain response is:

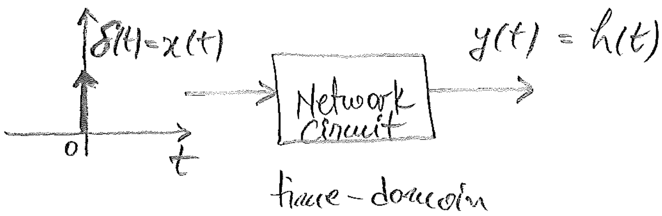
$$\boxed{Y(s) = H(s)\cdot X(s)} \quad (3)$$

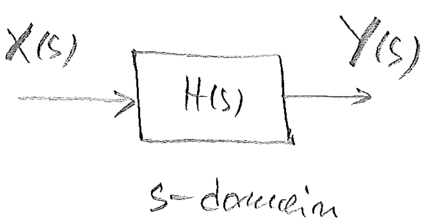
OBS. Zero-state is when reactive elements have zero initial conditions! and the $Y_{natural}(s)$ component of the complete response is zero!

- In this case, the forced response is:

$$\boxed{y(t) = \mathcal{L}^{-1}\{H(s) \cdot X(s)\}} \quad (4)$$

Special particular case: response to the unit impulse





$$Y(s) = H(s) \cdot X(s) = H(s) \cdot 1 = H(s)$$

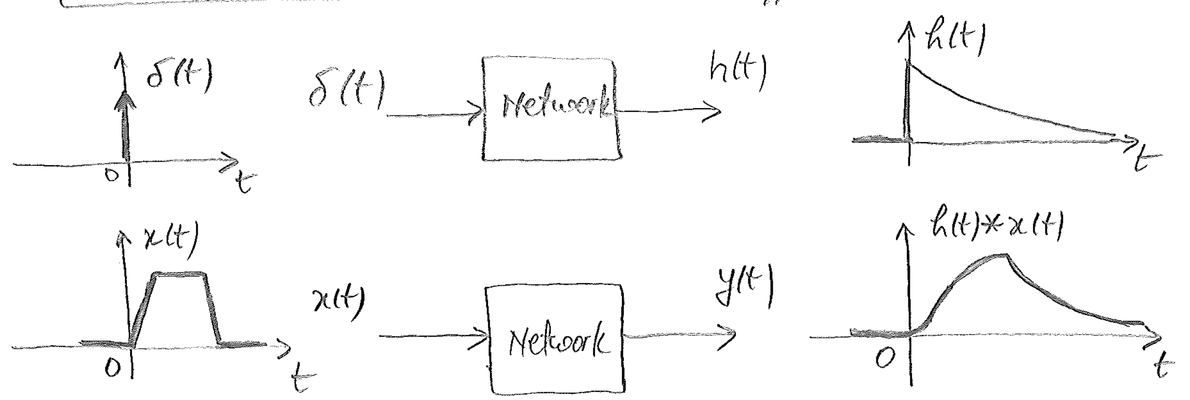
$y(t) = \mathcal{L}^{-1}\{H(s)\} = h(t)$: called the unit-impulse response function, or the $\boxed{\text{impulse response}}$

$$\boxed{y(t) = h(t)} \quad (5)$$

It is a very important descriptive property of a circuit.

That is because if we know $h(t)$, then, we can compute the response of the Network to a new forcing function $x(t)$ as:

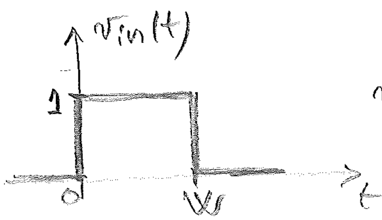
$$\boxed{y(t) = h(t) * x(t) = x(t) * h(t)} \quad (6)$$

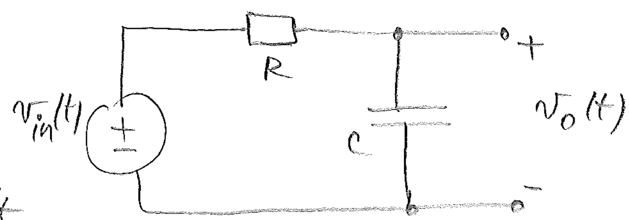


Convolve $h(t)$ with $x(t)$ to find $y(t)$!

- This is especially usefull when for example the applied signal $x(t)$ or the impulse-response $h(t)$ are known only through <u>experimental data</u> and may thus lack <u>explicit</u> Laplace transforms. In such cases, it is <u>easier</u> to do the <u>convolution</u> operation in the time domain!

$\boxed{\text{Example 1:}}$ Find the response of the RC circuit to a unity pulse of width W. Use convolution!
(<u>Note</u>: See also example #2 from lecture week #8, Mon)





Applied input: $v_{in}(t) = u(t) - u(t-W)$ [V]

impulse response: $\boxed{h(t) = \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}}}$ [V] where $\tau = RC$

prove it!

Use convolution:

$$v_o(t) = v_{in}(t) * h(t) = \int_0^t v_{in}(\xi) \cdot h(t-\xi)\, d\xi$$

<u>case 1</u>: $\boxed{t \le 0}$, $v_o(t) = 0$ because $v_{in}(t)$ and $h(t)$ are <u>causal</u>!

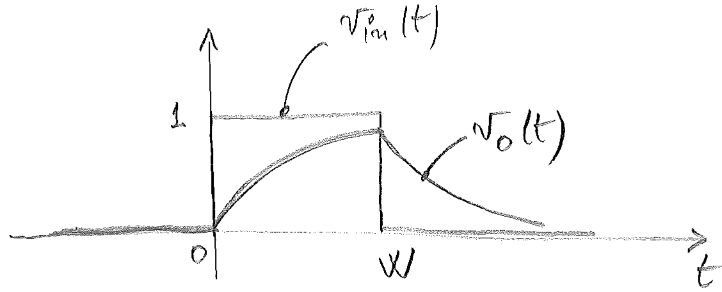
<u>Case 2</u>: $\boxed{0 \le t \le W}$

$$v_o(t) = \int_0^t 1 \times \frac{1}{\tau} e^{-(t-\xi)/\tau}\, d\xi = \frac{1}{\tau} e^{-\frac{t}{\tau}} \cdot \left(\tau \cdot e^{\xi/\tau} \Big|_0^t \right) =$$

$$v_o(t) = 1 - e^{-\frac{t}{\tau}} \quad [V]$$

<u>Case 3</u>: $\boxed{t \ge W}$

$$v_o(t) = \int_0^W 1 \times e^{-(t-\xi)/\tau}\, d\xi = \frac{1}{\tau} e^{-\frac{t}{\tau}} \left(\tau e^{\frac{\xi}{\tau}} \Big|_0^W \right) = \left(e^{\frac{W}{\tau}} - 1 \right) e^{-\frac{t}{\tau}} \quad [V]$$



$v_{in}(t)$
1
$v_0(t)$
0
W
t