6 Repeated real poles

- Consider one simple pole at $s=p$ with *multiplicity* r

$$D(s) = (s-p)^r$$

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p)^r} = \boxed{\frac{A_r}{(s-p)^r} + \frac{A_{r-1}}{(s-p)^{r-1}} + \dots + \frac{A_1}{(s-p)}} \quad (11)$$

partial fraction expansion contains all powers of $s-p$ from r to 1.

- Again, if $A_r, A_{r-1}, \dots, A_1$ are known, then using:

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^k}\right\} = \frac{t^{k-1} e^{-at}}{(k-1)!} \cdot u(t)} \quad (12)$$

we can write:

$$\boxed{f(t) = \left[\frac{A_r}{(r-1)!} \cdot t^{r-1} + \dots + A_3 t^2 + A_2 t + A_1\right] e^{pt} \cdot u(t)} \quad (13)$$

where A_r thru A_1 can be found by:

$$\boxed{A_i = \frac{1}{(r-i)!} \cdot \frac{d^{r-i}}{ds^{r-i}} (s-p)^r \cdot F(s) \Big|_{s=p}} \quad (14')$$

for: $i = 1, 2, \dots, r-1$

and for: $i = r$:

$$\boxed{A_r = (s-p)^r \cdot F(s) \Big|_{s=p}} \quad (14)$$

Example: Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$, $F(s) = \dfrac{8(s+2)}{(s+1)^3 (s+3)}$

Repeated real pole: $p_1 = -1$ Np/s with multiplicity 3
Real simple pole: $p_2 = -3$ Np/s.

$$F(s) = \frac{A_{1,3}}{(s+1)^3} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{1,1}}{s+1} + \frac{A_2}{s+3}$$

$$A_{1,3} = (s+1)^3 \cdot F(s)\Big|_{s=-1} = \frac{8(s+2)}{s+3}\Big|_{s=-1} = 4$$

$$A_{1,2} = \frac{d}{ds}\,\frac{8(s+2)}{(s+3)}\Big|_{s=-1} = \frac{8}{(s+3)^2}\Big|_{s=-1} = 2$$

$$A_{1,1} = \frac{1}{2}\frac{d}{ds}\,\frac{8}{(s+3)}\Big|_{s=-1} = \frac{1}{2}\cdot\frac{-16}{(s+3)^3}\Big|_{s=-1} = -1$$

$$A_2 = (s+3)\cdot F(s)\Big|_{s=-3} = \frac{8(s+2)}{(s+1)^3}\Big|_{s=-3} = 1$$

$\Rightarrow$

$$\Rightarrow F(s) = \frac{4}{(s+1)^3} + \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+3}$$

$$\Rightarrow \boxed{f(t) = \left[\left(4t^2+2t-1\right)e^{-t} + e^{-3t}\right]\cdot u(t)}$$

C Complex-conjugate poles

if $F(s)$ has as complex-conjugate pole pair:

$$P_1 = \alpha + j\beta$$
$$P_2 = \alpha - j\beta$$

then $D(s)$ contains the quadratic term: $(s-P_1)(s-P_2) = (s-\alpha)^2+\beta^2$

$$F(s) = \frac{N(s)}{(s-\alpha)^2+\beta^2} = \frac{N(s)}{(s-\alpha-j\beta)(s-\alpha+j\beta)} = \frac{\mathbb{A}_1}{s-\alpha-j\beta} + \frac{\mathbb{A}_2}{s-\alpha+j\beta}$$

bold face because they are complex!!!

Like before:

$$\mathbb{A}_1 = (s-\alpha-j\beta)\cdot F(s)\Big|_{s=\alpha+j\beta} = \frac{N(\alpha+j\beta)}{2j\beta}$$

$$\mathbb{A}_2 = (s-\alpha+j\beta)\cdot F(s)\Big|_{s=\alpha-j\beta} = \frac{N(\alpha-j\beta)}{-2j\beta} = \mathbb{A}_1^* \leftarrow \text{conjugate of } \mathbb{A}_1$$

- So, we need to compute only <u>one</u> residue [the residue at the <u>upper</u> pole in the s-plane): $p_1 = \alpha + j\beta$:

$$\boxed{A_1 = (s - \alpha - j\beta) \cdot F(s) \Big|_{s = \alpha + j\beta}} \quad (15)$$

- Letting $A_1 = |A_1| \cdot \angle\, \measuredangle A_1$

$A_2 = |A_1| \cdot \angle -\measuredangle A_1$ use equation (9) to write:

$$f(t) = \left[|A_1| \cdot e^{j\measuredangle A_1} \cdot e^{(\alpha + j\beta)t} + |A_1| \cdot e^{-j\measuredangle A_1} \cdot e^{(\alpha - j\beta)t} \right] \cdot u(t)$$

$$= |A_1| \cdot e^{\alpha t} \cdot \left[e^{j(\beta t + \measuredangle A_1)} + e^{-j(\beta t + \measuredangle A_1)} \right] \cdot u(t)$$

also using: $\boxed{e^{j\theta} + e^{-j\theta} = 2\cos\theta}$ $\Rightarrow$

$$\Rightarrow \boxed{f(t) = \mathcal{L}^{-1}\left\{ \frac{A_1}{s - \alpha - j\beta} + \frac{A_1^*}{s - \alpha + j\beta} \right\} = 2|A_1| \cdot e^{\alpha t} \cdot \left[\cos(\beta t + \measuredangle A_1) \right] \cdot u(t)} \quad (16)$$

As we expected: A damped sinusoid!

- $\alpha \equiv$ negative of the damping coeff.
- $\beta \equiv$ angular frequency
- $2|A_1| \equiv$ amplitude
- $\measuredangle A_1 \equiv$ phase angle

<u>Note</u>: (16) can be generalized to any number of complex conjugate pole pairs!

$\boxed{\text{Example}}$: Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$, $F(s) = \dfrac{40(s+1)}{(s+5)(s^2+6s+25)}$

One real pole $p_1 = -5$ Np/s

Two complex-conjugate poles: $\begin{cases} p_2 = -3+j4 & \text{Np/s} \\ p_3 = -3-j4 & \text{Np/s} \end{cases}$

$$F(s) = \frac{40(s+1)}{(s+5)(s+3-j4)(s+3+j4)} = \frac{\overset{\text{real}}{A_1}}{s+5} + \frac{A_2}{s+3-j4} + \frac{A_2^*}{s+3+j4}$$

complex

$$A_1 = (s+5)F(s)\Big|_{s=-5} = \frac{40(s+1)}{s^2+6s+25}\Big|_{s=-5} = -8$$

$$A_2 = (s+3-j4)\cdot F(s)\Big|_{s=-3+j4} = \frac{40(s+1)}{(s+5)(s+3+j4)}\Big|_{s=-3+j4} = \frac{40(-3+j4+1)}{(-3+j4+5)\,j8} =$$

$$= 5\cdot\frac{1-j2}{2-j1} = 5\angle -36.87^\circ$$

Hence, $\boxed{f(t) = \left[-8e^{-5t} + 10\cdot e^{-3t}\cdot\cos(4t-36.87^\circ)\right]\cdot u(t)}$

TO DO: checks with initial/final value theorems!

$\boxed{d}$ Repeated complex pole pairs

- Are treated in similar fashion, except that the calculations involve complex algebra!

$\boxed{\text{Example}}$: Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ where $F(s) = \dfrac{2}{(s^2+4s+5)^2}$

Repeated pole pair: $p_{1,2} = -2 \pm j1$ complex Np/s with multiplicity 2

$$F(s) = \frac{2}{(s+2-j1)^2(s+2+j1)^2} =$$

$$= \frac{A_{1,2}}{(s+2-j1)^2} + \frac{A_{1,1}}{(s+2-j1)} + \frac{A_{1,2}^*}{(s+2+j1)^2} + \frac{A_{1,1}^*}{(s+j+j1)}$$

Residues:

$$A_{1,2} = (s+2-j1)^2 \cdot F(s)\Big|_{s=-2+j1} = \frac{2}{(s+2+j1)^2}\Big|_{s=-2+j1} = -0.5$$

$$A_{1,1} = \frac{d}{ds}\frac{2}{(s+2+j1)^2}\Big|_{s=-2+j1} = \frac{-2}{(s+2+j1)^3}\Big|_{s=-2+j1} = 0.5\,\angle{-90^\circ}$$

Hence,

$$f(t) = \left[-t\cdot e^{-2t}\cdot \cos t + e^{-2t}\cdot \cos(t-90^\circ)\right]\cdot u(t)$$

$$\boxed{f(t) = (\sin t - t\cos t)\cdot e^{-2t}\cdot u(t)}$$

e) Improper rational functions

if $m \geq n$, $F(s)$ is improper and we must perform long division to put it in form:

$$\boxed{F(s) = \frac{N(s)}{D(s)} = \underbrace{Q(s)}_{\text{quotient}} + \underbrace{\frac{\overbrace{R(s)}^{\text{remainder}}}{D(s)}}_{\text{now, this is proper!}}} \qquad (17)$$

Example:

Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ where $F(s) = \dfrac{2s^3+11s^2+14s-5}{s^2+5s+6}$

$m=3$, $n=2$ $\Rightarrow$ $F(s)$ is improper! Do long division:

$$\begin{array}{r|l}
\cancel{2s^3}+11s^2+14s-5 & s^2+5s+6 \\
\underline{\cancel{2s^3}+10s^2+12s} & 2s+1 \leftarrow Q(s) \\
\cancel{s^2}+2s-5 & \\
\underline{\cancel{s^2}+5s+6} & \\
-3s-11 \leftarrow R(s) &
\end{array}$$

$$F(s) = 2s+1+\frac{-3s-11}{s^2+5s+6} = 2s+1+\frac{2}{s+3}-\frac{5}{s+2}$$

$$\Rightarrow \boxed{f(t) = 2\frac{d}{dt}\delta(t) + \delta(t) + (2e^{-3t}-5e^{-2t})\cdot u(t)}$$