Continued from last time!

8. $\boxed{\text{Initial and final values}}$

- Given the pair: $f(t) \Leftrightarrow F(s)$
  we are interested in finding $f(0^+) = \lim_{t \to 0^+} f(t)$ and $f(\infty) = \lim_{t\to\infty} f(t)$
  once $F(s)$ is known.
- These limiting values provide useful checks for our calculations!

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \boxed{sF(s) - f(0^-)}$$

Skip this in class

$$= \int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt = \int_{0^-}^{0^+} \frac{df}{dt} e^{s0} dt + \int_{0^+}^{+\infty} \frac{df}{dt} e^{-st} dt$$

$$= \int_{0^-}^{0^+} df + \int_{0^+}^{+\infty} \frac{df}{dt} e^{-st} dt = \boxed{f(0^+) - f(0^-) + \int_{0^+}^{+\infty} \frac{df}{dt} e^{-st} dt}$$

$$\Rightarrow \boxed{sF(s) = f(0^+) + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st} dt} \quad (1)$$

- Now, let $s \to \infty$ in (1) to get: $\lim_{s\to\infty} f(0^+) = \boxed{f(0^+) = \lim_{s\to\infty} sF(s)}$ (2)

a result referred to as "initial value theorem"!

- Now, let $s \to 0$ to get:

$$\lim_{s\to 0}\left[sF(s) - f(0^-)\right] = \lim_{s \to 0}\left[\mathcal{L}\left\{\frac{df}{dt}\right\}\right] = \lim_{s\to 0} \int_{0^-}^{+\infty} \frac{df}{dt} e^{-st} dt =$$

$$= \int_{0^-}^{+\infty} \frac{df}{dt} e^{0t} dt = \int_{0^-}^{+\infty} df = f(+\infty) - f(0^-) \quad (3)$$

- From (3), where $\lim_{s\to 0} f(0^-) = f(0^-)$, we get:

$$\boxed{f(+\infty) = \lim_{s\to 0} s \cdot F(s)} \quad (4)$$

referred to as the "final value theorem"!

Example

Find Laplace transform of: $f(t) = (5 + 6te^{-3t} - 4e^{-t}\cos 2t)\cdot u(t)$
then use initial and final value theorems to check your results!

- Use Laplace transform pairs (last lecture + table from textbook) to find:

$$F(s) = \underset{\substack{\uparrow \\ \text{by linearity}}}{\frac{5}{s}} + \frac{6}{(s+3)^2} - \frac{4(s+1)}{(s+1)^2 + 2^2}$$

$$\Rightarrow sF(s) = \frac{s^4 + 18s^3 + 82s^2 + 234s + 225}{s^4 + 8s^3 + 26s^2 + 48s + 45}$$

- Check: $f(0^+) = 5 + 6\times 0 - 4e^0 \cos 0 = 1$
  $\lim_{s\to\infty} sF(s) = \frac{s^4}{s^4} = 1$
  $\Big\} \Rightarrow f(0^+) = \lim_{s\to\infty} sF(s) = 1$

- check: $f(+\infty) = 5 + 0 + 0 = 5$
  $\lim_{s\to 0} sF(s) = \frac{225}{45} = 5$
  $\Big\} \Rightarrow f(\infty) = \lim_{s\to 0} sF(s) = 5$

both checks are in agreement!

5 The inverse Laplace transform

- In circuit analysis, when applying Laplace transform, we may get rational functions of s, expressed as ratios of two polynomials:

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_0} \tag{5}$$

roots are zeros (numerator); roots are poles (denominator)

- Coeff: $\left\{\begin{array}{l} a_i,\ i = 1,2,\dots,m \\ b_k,\ k = 1,2,\dots,n \end{array}\right\}$ are real.
- Roots of $D(s) = 0$ are called the $\boxed{\text{poles}}$ of $F(s)$
- Roots of $N(s) = 0$ are called the $\boxed{\text{zeros}}$ of $F(s)$

- If $b_k \geq 0$, $k = 1, 2, \dots, n$ then, the poles lie in the left half of the $\boxed{\text{s plane}}$ or at most on the imaginary axis.

OBS: we'll revisit this later!

- $F(s)$ is said to be $\boxed{\text{proper}}$ if $m < n$ and to be $\boxed{\text{improper}}$ if $m \geq n$.

- Our goal in this section is to find:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = ?$$

from eq. (5).

$\boxed{a}$ $\boxed{\text{Real and distinct poles}}$

- In this case denominator can be factored as:

$$\boxed{F(s) = \frac{N(s)}{(s-p_1)(s-p_2)\dots(s-p_n)}} \quad (6)$$

$p_1, p_2, \dots, p_n$ are the poles.

- If $F(s)$ is $\boxed{\text{proper}}$, then it's possible to express it as a "partial fraction expansion"

$$\boxed{F(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \dots + \frac{A_n}{s-p_n}} \quad (7)$$

$A_1, A_2, \dots, A_n$ are suitable coefficients called the $\boxed{\text{residues}}$ of $F(s)$

- If we knew A_k, $k = \overline{1,n}$ we could exploit linearity to with property: $\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \cdot u(t)}$ (8)

to get:

$$\boxed{f(t) = \left(A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t}\right) \cdot u(t)} \quad (9)$$

where residues can be found as: $\boxed{A_k = (s-p_k) \cdot F(s)\Big|_{s=p_k}}$ (10)

or: "residue A_k of $F(s)$ at a simple pole real p_k is found by taking the product $(s-p_k)F(s)$ and evaluating it at $s = p_k$"!!!

$\boxed{\text{Example}}$ Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$, $F(s) = \dfrac{2s^2+s-3}{s(s^2+4s+3)}$

$D(s) = s(s^2+4s+3)$ has simple real roots, which are the poles:

$$P_1 = 0, \quad P_2 = -1, \quad P_3 = -3 \quad \text{i.e., } D(s) = s(s+1)(s+3)$$

$m = 2$, $n = 3$ $\Rightarrow$ $F(s)$ is proper! so we can express it as:

$$F(s) = \frac{2s^2+s-3}{s(s+1)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+3}$$

$$A_1 = sF(s)\Big|_{s=0} = \frac{2s^2+s-3}{(s+1)(s+3)}\Bigg|_{s=0} = \frac{2\times 0^2+0-3}{(0+1)(0+3)} = -1$$

$$A_2 = sF(s)\Big|_{s=-1} = \frac{2s^2+s-3}{s(s+3)}\Bigg|_{s=-1} = 1$$

$$A_3 = sF(s)\Big|_{s=-3} = \frac{2s^2+s-3}{s(s+1)}\Bigg|_{s=-3} = 2$$

$\Rightarrow$

$$\Rightarrow F(s) = -\frac{1}{s} + \frac{1}{s+1} + \frac{2}{s+3}$$

- Therefore: $f(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}$

$$f(t) = -1\cdot u(t) + 1e^{-t}\cdot u(t) + 2e^{-3t}\cdot u(t)$$

$$\boxed{f(t) = (-1 + e^{-t} + 2e^{-3t})\cdot u(t)}$$

- Final checks:

$$f(0^+) = 2 = \lim_{s\to\infty} sF(s) \ !$$

$$f(\infty) = -1 = \lim_{s\to 0} sF(s) \ !$$

which further corroborates our results!