4 Basic theorems for the Laplace transform (operational transforms)

1. Linearity

$$\boxed{\mathcal{L}\{a f_1(t) + b f_2(t)\} = a \cdot F_1(s) + b F_2(s)} \quad (1)$$

two causal functions : $f_1(t) \Leftrightarrow F_1(s)$; $f_2(t) \Leftrightarrow F_2(s)$
a, b = constants.

- The Laplace transform of a linear combination is the linear combination of the individual transforms.

$$\boxed{\mathcal{L}\{k \cdot f(t)\} = k \cdot F(s)} \quad (2)$$ particular case.

Example: $f(t) = 2(1 - 5e^{-3t}) \cdot u(t)$

$$\mathcal{L}\{f(t)\} = 2\left(\mathcal{L}\{u(t)\} - 5\mathcal{L}\{e^{-3t}u(t)\}\right) = 2\left(\frac{1}{s} - \frac{5}{s+3}\right) = \frac{-8s+6}{s(s+3)}$$

2. Differentiation

$$\boxed{\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)} \quad (3)$$

- Differentiation in the t domain corresponds to multiplication by s in the s domain, followed by subtraction of the initial value $f(0^-)$!
- Like in the case of phasors, the differentiation operation is changed into an algebraic operation!
- However, unlike the phasor, the Laplace transform accounts also for the initial value of the function!
- Generalization of (3):

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f^{(1)}(0^-) - \dots - f^{(n-1)}(0^-)} \quad (3')$$

where used notation: $f^{(n)} \triangleq \frac{d^n f}{dt^n}$

3. Integration

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\xi)\,d\xi\right\} = \frac{1}{s}\cdot F(s) \qquad (4)$$

dummy variable of integration

- Integration in the time domain corresponds to division by s in the s domain.

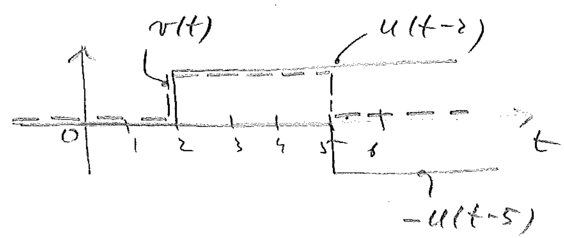
4. Time shifting

$$\mathcal{L}\{f(t-a)\cdot u(t-a)\} = e^{-as}\cdot F(s) \qquad (5)$$

- Shifting by $a>0$ in the t domain corresponds to multiplication by e^{-as} in the s domain.

Example: $v(t) = u(t-2) - u(t-5)$

$\mathcal{L}\{v(t)\} = ?$



$$\mathcal{L}\{u(t-2)-u(t-5)\} = e^{-2s}\cdot\underbrace{\mathcal{L}\{u(t)\}}_{=1/s} - e^{-5s}\cdot\mathcal{L}\{u(t)\} = \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s}$$

$$= \frac{e^{-2s}-e^{-5s}}{s}.$$

5. Frequency shifting

$$\mathcal{L}^{-1}\{F(s+a)\} = e^{at}\cdot f(t) \qquad (6)$$

- Shifting by $a>0$ in the s domain corresponds to multiplication by e^{-at} in the t domain! a is amount of shift in complex Np/s!

6. **Scaling**

$$\boxed{\mathcal{L}\{f(at)\} = \frac{1}{a}\cdot F\left(\frac{s}{a}\right)} \quad (7)$$

$$\boxed{\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a}\cdot f\left(\frac{t}{a}\right)} \quad (8)$$

- Scaling by $a>0$ in one domain corresponds to scaling by $\frac{1}{a}$ in the other domain, followed by multiplication by $1/a$.

7. **Convolution**

- Given $F(s)$ and $G(s)$ corresponding Laplace transforms to $f(t)\cdot u(t)$ and $g(t)\cdot u(t)$, we wish to find $\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = ?$

$$F(s)\cdot G(s) = \left(\int_{0^-}^{+\infty} f(\xi)e^{-s\xi}d\xi\right)\cdot G(s) = \int_{0^-}^{+\infty} f(\xi)\left[G(s)e^{-s\xi}\right]d\xi$$

$$= \int_{0^-}^{+\infty} f(\xi)\cdot \mathcal{L}\{g(t-\xi)\cdot u(t-\xi)\}d\xi$$

time shifting

$$= \int_{0^-}^{+\infty} f(\xi)\cdot\left(\int_0^{+\infty} g(t-\xi)\cdot u(t-\xi)e^{-st}dt\right)d\xi$$

interchange order of integration and $u(t-\xi)$ as function of ξ satisfies $u(t-\xi)=0$ for $\xi>t$, second integral upper limit is t

$$= \int_{0^-}^{\infty}\left(\int_{0^-}^{t} f(\xi)g(t-\xi)d\xi\right)dt$$

Define "convolution" of $f(t)$ and $g(t)$ as:

$$\boxed{f(t)*g(t) \triangleq \int_{0^-}^{t} f(\xi)g(t-\xi)\,d\xi} \quad (9)$$

we get:

$$\boxed{f(t) * g(t) \leftrightarrow F(s)\cdot G(s)} \quad (10)$$

- Convolution in the time domain corresponds to multiplication in the s domain.

- Convolution is commutative:

$$f(t) * g(t) = g(t) * f(t) \triangleq \int_{0^-}^{t} g(\xi)\cdot f(t-\xi)\,d\xi.$$

$\boxed{\text{Example}}$: Use convolution to find:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = ?$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F(s)}\cdot\underbrace{\frac{1}{s+2}}_{G(s)}\right\} = g(t) * f(t) = \int_{0^-}^{t} g(\xi)\cdot f(t-\xi)\,d\xi =$$

$$F(s) \leftrightarrow f(t) = u(t) \qquad G(s) \leftrightarrow e^{-2t}\cdot u(t) = g(t)$$

$$= \int_{0^-}^{t} e^{-2\xi}\cdot u(\xi)\cdot u(t-\xi)\,d\xi = \int_{0^-}^{t} e^{-2\xi}\,d\xi = \frac{1}{2}\left(1 - e^{-2t}\right)\cdot u(t).$$