2 The **Laplace** Transform ← v. v. important!

Remember: we switched from an ac signal $v(t) = V\cos(\omega t + \theta)$ to its phasor $\mathbb{V}(\omega) = V_m \angle\theta$; this was in effect a transformation from the time domain to a function of frequency:

$$v(t) \rightarrow \mathbb{V}(\omega)$$

Likewise, when we switched back from a phasor to its corresponding ac signal we performed the inverse transformation:

$$\mathbb{V}(\omega) \rightarrow v(t).$$

Now, the Laplace transform, effects a transformation from functions of time t to functions of the complex frequency s, $f(t) \rightarrow F(s)$ according to:

$$\boxed{F(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-st}\, dt} \quad (1) \qquad \boxed{s = \sigma + j\omega} \quad (2) \; [\mathrm{Np/s}]$$

↑ referred to as double-sided or bilateral Laplace transform because the integration limits are $-\infty$ and $+\infty$.

- Physically realizable circuits do not respond to a signal until it is actually applied ⇒ referred as causal systems (effect never precedes the cause)
- It is convenient to take $t = 0$ as the instant at which signal is applied. and the response manifests itself only for $t \geq 0$. ⇒ causal signals or positive-time signals!
- Hence, we can assume the lower limit of integration 0 and make the transform single sided. Actually it is made 0^- (to also include the impulse function $\delta(t)$ and higher-order singularities at the origin)

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{+\infty} f(t)\, e^{-st}\, dt} \quad (3)$$

OBS: whatever happened before $t=0^-$ is accounted for by the initial conditions, which are memorized by the circuit in its energy-storage elements!

- $f(t)$ is said to be the inverse Laplace transform of $F(s)$:

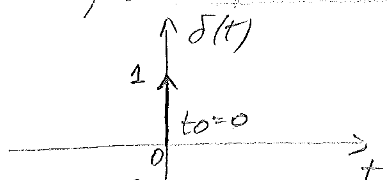
$$\boxed{f(t)=\mathcal{L}^{-1}\{F(s)\}} \quad (4) \quad = \underbrace{\frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s)e^{st}\,dt}_{\text{not really used.}}$$

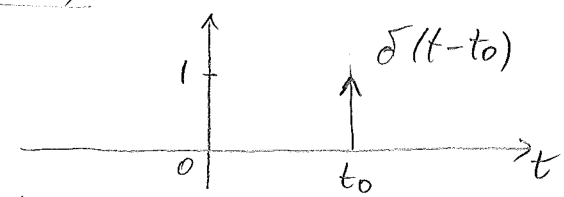
- $f(t)$ and $F(s)$ form a transform pair:

$$\boxed{f(t) \Leftrightarrow F(s)} \quad (5)$$

3 Transform pairs

- Unit-impulse function: $\delta(t)$ — Dirac function.





Defined to have an area of unity: $\delta(t-t_0)=0$ for $t\neq t_0$ and $\int_{t_0-\varepsilon}^{t_0+\varepsilon}\delta(t-t_0)\,dt=1$

$t_0=0$:
$$\mathcal{L}\{\delta(t)\} = \int_{-0}^{\infty}\delta(t)e^{-st}\,dt = e^{-s0} = 1$$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_{-0}^{+\infty}\delta(t-t_0)e^{-st}\,dt = e^{-st_0}$$

$$\boxed{\delta(t-t_0) \Leftrightarrow e^{-st_0}} \quad (6)$$

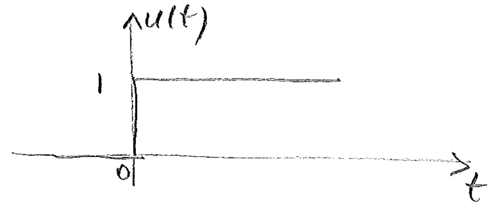
$$\boxed{\delta(t) \Leftrightarrow 1} \quad (6')$$

Sifting property or sampling property

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)\,dt = f(t_0)$$

$t_1<t_0<t_2$; f continuous at t_0.

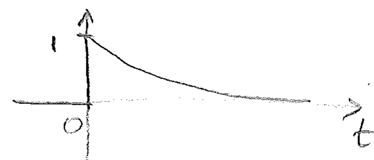
- $\boxed{\text{The unit step function}}$: $\boxed{u(t)}$



$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0^-}^{\infty} 1 \cdot e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{0^-}^{\infty} = \frac{1}{s}$$

$$\boxed{u(t) \Leftrightarrow \frac{1}{s}} \quad (7)$$

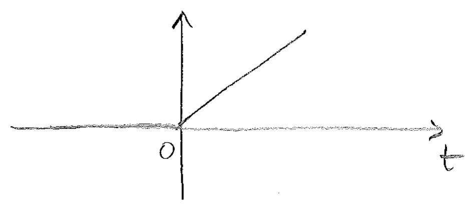
- $\boxed{\text{(Decaying) exponential}}$: $\boxed{e^{-at} \cdot u(t)}$



$$\mathcal{L}\{e^{-at}\} = \int_{0^-}^{+\infty} e^{-at} \cdot e^{-st} dt = \int_{0}^{+\infty} e^{-(s+a)t} dt = -\frac{1}{s+a} e^{-(s+a)t} \Big|_{0^-}^{+\infty} = \frac{1}{s+a}$$

$$\boxed{e^{-at} \cdot u(t) \Leftrightarrow \frac{1}{s+a}} \quad (8)$$

- $\boxed{\text{Ramp function}}$: $\boxed{t \cdot u(t)}$

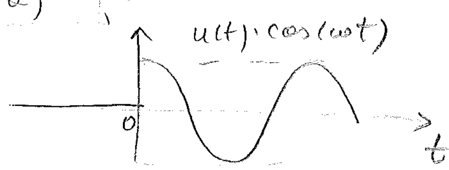


$$\mathcal{L}\{t \cdot u(t)\} = \int_{0^-}^{+\infty} t e^{-st} dt = \frac{1}{s^2}$$

$$\boxed{t\, u(t) \Leftrightarrow \frac{1}{s^2}} \quad (9)$$

Also: $\boxed{t \cdot e^{-at} \cdot u(t) \Leftrightarrow \frac{1}{(s+a)^2}}$ (10)

- $\boxed{\text{Cosine function}}$: $u(t) \cdot \cos \omega t$



$$\mathcal{L}\{\cos \omega t\} = \int_{0^-}^{\infty} \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) e^{-st} dt \underset{(8)}{=} \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

$$\boxed{u(t) \cdot \cos \omega t \Leftrightarrow \frac{s}{s^2+\omega^2}} \quad (11)$$