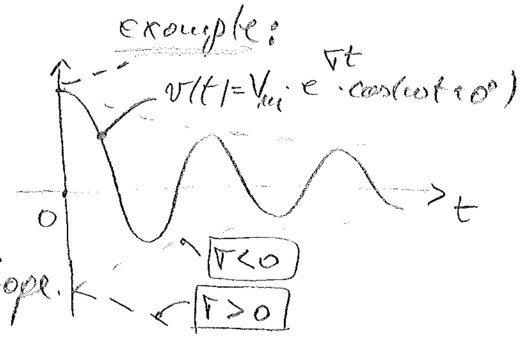
Chapter 14

Complex frequency and the Laplace transform

1 Complex frequency

- Consider an exponentially dumped sinusoidal function:

(1) $\boxed{v(t) = V_m \cdot e^{\sigma t} \cdot \cos(\omega t + \theta)}$



let $\begin{cases} \sigma = 0 \\ \omega = 0 \end{cases}$

(2) $\boxed{v(t) = V_m \cos\theta = V_0}$ a dc voltage.

$\sigma = 0$

(3) $\boxed{v(t) = V_m \cos(\omega t + \theta)}$ an ac voltage.

$\omega = 0$

(4) $\boxed{v(t) = V_m \cdot e^{\sigma t} \cdot \cos\theta = V_0 \cdot e^{\sigma t}}$ the exponential voltage.

all special cases of (1)

- Remember the complex representation of a sinusoidal function with phase-angle zero:

(5) $\boxed{v(t) = V_0 \cdot e^{j\omega t}}$

- Similarity between (4) & (5) is emphasized by describing σ as "frequency". More precisely as the "real part" of the complex frequency.

defined next page.

General form

- A "complex exponential signal" is a signal of the form:

$$\boxed{\mathbf{x}(t) = \mathbf{X} \cdot e^{st}} \quad (6)$$

bold

Note: the 1st time we mix phasors and time variable!

- $\mathbf{X}$, s are "time-independent complex parameters" expressed in polar and rectangular coordinates as:

Note, it's a phasor!

$$\boxed{\begin{aligned} \mathbf{X} &= X_m \cdot e^{j\theta} \\ s &= \sigma + j\omega \end{aligned}} \quad (7)$$

phase angle [radians or degrees]

magnitude of $x(t)$ may be [V or A.]

Note: use $x(t)$ because in general can be a voltage or current!

real part of s $\triangleq$ Neper frequency in [nepers/s]

imaginary part of s; called radian frequency [radians/s]

$\triangleq$ complex frequency

$$\Rightarrow \boxed{\mathbf{x}(t) = X_m \cdot e^{j\theta} \cdot e^{(\sigma + j\omega)t} = X_m \cdot e^{\sigma t} \cdot e^{j(\omega t + \theta)}} \quad (6')$$

- Using Euler's formula, we have the following alternate form for a complex exponential signal:

Note: this is a complex function!

$$\boxed{\mathbf{x}(t) = X_m \cdot e^{\sigma t} \cdot \left[\cos(\omega t + \theta) + j \sin(\omega t + \theta) \right]} \quad (8)$$

Because it's complex, it cannot be duplicated in laboratory, where signals are real. However, mathematicians showed that real-world signals can be expressed as linear combinations of complex exponential signals. Knowing the response of a circuit to individual complex exponential signals, we can apply the "superposition principle".

Examples:

- Generally: $x(t)$ can be written as:
a real:

$$x(t) = \frac{1}{2}\cdot(x + x^{*}) = Re\{x\}$$

real function we are interested to use in circuits
Not bold character.

complex variables/functions.

complex exponential signal from equation 8)

its conjugate.

should be written with bold characters to show that they are complex.

- A real voltage signal can be written as:

$$v(t) = Re\{\underbrace{V_m \cdot e^{\sigma t} \cdot e^{j(\omega t+\theta)}}_{\text{complex exponential signal.}}\} = V_m e^{\sigma t}\cdot\cos(\omega t+\theta)$$

$v(t)$: real or time-domain

$V_m e^{\sigma t}\cdot\cos(\omega t+\theta)$: a exponentially varying sinusoidal

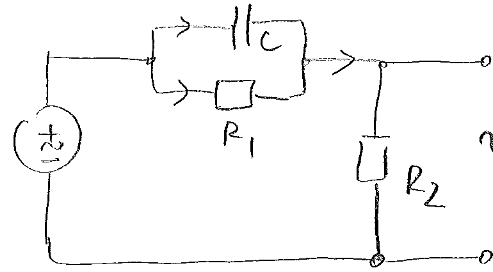
$\boxed{\text{Remember}}$: how we developed the frequency-domain descriptions of ac circuits, when we introduced the complex forcing function [$\cos\omega t = Re(e^{j\omega t})$ and we worked with complex source $V_m e^{j\omega t}$; response was complex whose real part corresponded to the real part of the complex source!] NOTE: Moreover, we have introduced phasors, which "carry" info on amplitude/magnitude and phase; ω was implicit/understood.

OBS: Here, we simply extended the notations to include the damped sinusoidal forcing function at a complex frequency. The difference is that here we replaced $\boxed{j\omega}$ with $\boxed{s}$!

Example: ① To develop a quick *feel* for how to handle complex exponential signals:

complex! (bold font)

$\mathbf{v}_i(t) = \mathbf{V}_i e^{st}$



$\mathbf{v}_0(t)$: complex! (bold font)

Note: this example may become confusing due to $v(t)$ time domain and $\mathbf{v}(t)$ complex!?

$\mathbf{v}_0(t) = \mathbf{V}_0 \cdot e^{st}$ and find relationship between $\mathbf{v}_i(t)$ and $\mathbf{v}_0(t)$

here these are complex variables

Solution:

KCL: $C\dfrac{d(\mathbf{v}_i-\mathbf{v}_0)}{dt} + \dfrac{\mathbf{v}_i-\mathbf{v}_0}{R_1} = \dfrac{\mathbf{v}_0}{R_2}$

$$\left.\begin{array}{l} R_1R_2C\dfrac{d\mathbf{v}_0}{dt} + (R_1+R_2)\mathbf{v}_0 = R_1R_2C\dfrac{d\mathbf{v}_i}{dt} + R_2\mathbf{v}_i \\ \mathbf{v}_i = \mathbf{V}_i e^{st} \\ \mathbf{v}_0 = \mathbf{V}_0 \cdot e^{st} \end{array}\right\} \Rightarrow$$

$$\Rightarrow R_1R_2Cs\,\mathbf{V}_0e^{st} + (R_1+R_2)\mathbf{V}_0e^{st} = R_1R_2Cs\,\mathbf{V}_i e^{st} + R_2\mathbf{V}_i e^{st}$$

$$\Rightarrow \boxed{\mathbf{v}_0(t) = \frac{R_1R_2Cs + R_2}{R_1R_2Cs + (R_1+R_2)} \cdot \mathbf{v}_i(t)}$$

Example: ② In previous example for $R_1 = 1\Omega$, $R_2 = 2\Omega$, $C = 1F$, given:

$v_i(t) = 10 \cdot e^{-0.5t} \cdot \cos(1.5t + 30°)\,V$ find: $v_0(t) = ?$

$$v_i(t) = Re\{\mathbf{v}_i(t)\} = Re\{10\angle 30° \cdot e^{(-0.5+j1.5)t}\}$$

real time-domain

complex exponential signal

should be understood from context.

The complex response $\mathbf{v}_o(t)$ to the complex signal $\mathbf{v}_i(t)$ is:

$$\mathbf{v}_o(t) = \frac{2(-0.5+j1.5)+2}{2(-0.5+j1.5)+3} \cdot 10\angle 30^\circ \cdot e^{(-0.5+j1.5)t}$$

$$= 8.771 \angle 45.26 \cdot e^{(-0.5+j1.5)t} \quad V$$

The real response then $v_o(t)$ to the real signal $v_i(t)$ is:

that is time domain $v_o(t) = Re\{\mathbf{v}_o(t)\}$

or: $\boxed{v_o(t) = 8.771 \cdot e^{-0.5t} \cdot \cos(1.5t + 45.26^\circ)}$ V

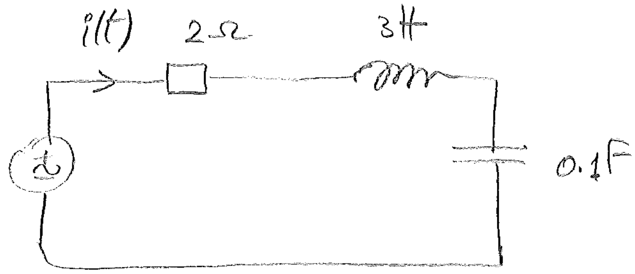
which is a damped sinusoid with the same damping factor σ and frequency ω as the applied signal, differing only in amplitude and phase!

Example 3

Given:

$v(t) = 60 \cdot e^{-2t} \cdot \cos(4t + 10^\circ)$

$\uparrow V_m \quad \uparrow \sigma \quad \omega \quad \theta$



"time-domain" circuit representation.

Find: $i(t) = \underset{?}{I_m} \, e^{-2t} \cdot \cos(4t + \underset{?}{\phi})$

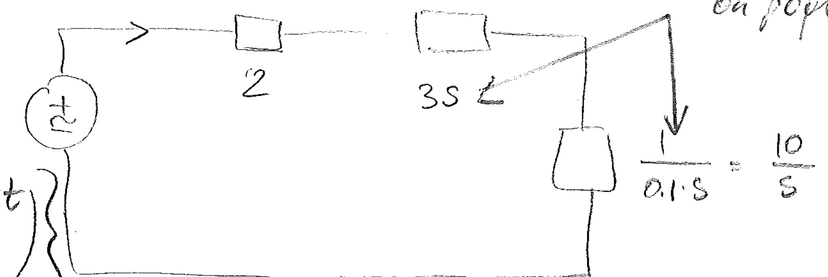
We expect: $\mathbb{I} e^{st}$ ⇓

based on observation on page 3!

Use equation (6) → $\mathbb{V} e^{st}$

$$v(t) = \mathrm{Re}\left\{ (60 \angle 10^\circ) \cdot \left(e^{(-2+j4)t}\right) \right\}$$

$\boxed{s = -2 + j4}$



"complex frequency domain" representation

- Apply KVL:

$$60 \angle 10^\circ e^{st} = 2\,\mathbb{I} e^{st} + 3s\,\mathbb{I} e^{st} + \frac{10}{s}\,\mathbb{I} e^{st}$$

$$60 \angle 10^\circ = 2\,\mathbb{I} + 3s\,\mathbb{I} + \frac{10}{s}\,\mathbb{I}$$

$$\mathbb{I} = \left. \frac{60 \angle 10^\circ}{2 + 3s + 10/s} \right|_{s = -2+j4} \Rightarrow \mathbb{I} = \frac{60 \angle 10^\circ}{2 + 3(-2+j4) + 10/(-2+j4)} = 5.37 \angle -106.6^\circ$$

Therefore: $I_m = 5.37$ A, $\phi = -106.6^\circ$ and the time-domain real forced response is:

$$\boxed{i(t) = 5.37 \cdot e^{-2t} \cdot \cos(4t - 106.6^\circ)} \text{ A}$$