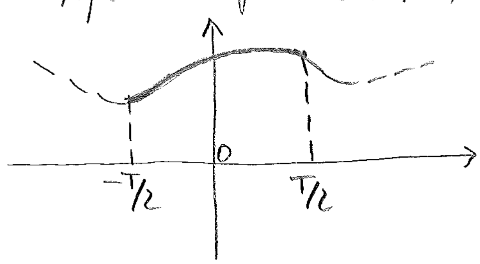
2 The Fourier transform

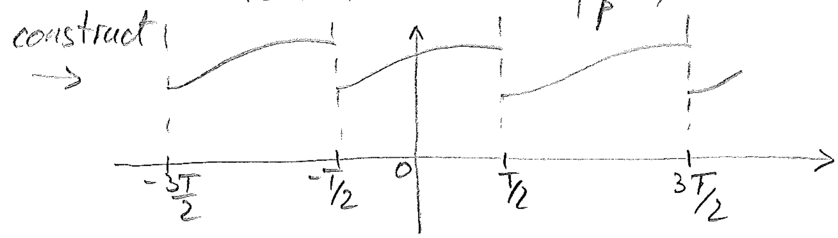
When a function is not periodic, we cannot find a Fourier series.

- In this case we may be able to find a $\boxed{\text{Fourier transform}}$ $\boxed{F(j\omega)}$, as a function of the continuous frequency ω that corresponds to the time-domain function $\boxed{f(t)}$.
- This transform plays the role for nonperiodic functions that phasors play for sinusoids, and that network functions may be obtained for transforms that are identical to those obtained for phasors.

Aperiodic function $f(t)$



Periodic extension $f_p(t)$



$f_p(t) = f_p(t+T)$ coincides with $f(t)$ over arbitrary interval:

$$f_p(t) = f(t), \quad -\frac{T}{2} \le t \le \frac{T}{2}$$

$\boxed{\text{idea}}$: Consider the exponential Fourier series of $f_p(t)$ and examine it in the limit $T \to \infty$, when clearly:

$$\lim_{T\to\infty} f_p(t) = f(t), \quad -\infty \le t \le +\infty$$

- By this process, we extend the Fourier series concept to the aperiodic function $f(t)$ by regarding it as periodic, but with an infinite period!
- This way, we can derive (see textbook):

$$(1) \quad \boxed{F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\cdot e^{-j\omega t}\, dt} \triangleq \text{Fourier transform of } f(t).$$

- $f(t)$ is the $\boxed{\text{inverse Fourier transform}}$ of $F(j\omega)$:

$$\boxed{f(t) = \mathcal{F}^{-1}\{F(j\omega)\}} \quad (2)$$

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega t}\, d\omega} \quad (3)$$

- $f(t)$ and $F(j\omega)$ are said to constitute a $\boxed{\text{transform pair}}$:

$$\boxed{f(t) \leftrightarrow F(j\omega)} \quad (4)$$

Example

$f(t) = e^{-at}\cdot u(t)$, $a>0$

- By definition:

$$F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} e^{-at}\cdot u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$

- Fourier transform pair:

$$\boxed{e^{-at}\cdot u(t) \leftrightarrow \frac{1}{a+j\omega}}$$

Fourier transform poirs

$\boxed{\delta(t) \Leftrightarrow 1}$ impulse function

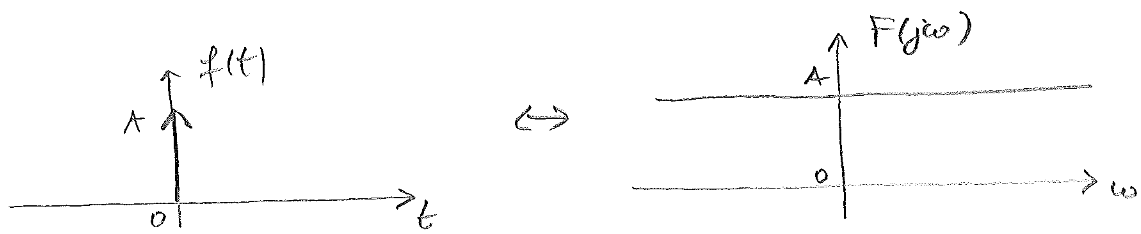
$\boxed{A\cdot\delta(t) \Leftrightarrow A}$ impulse of strength A

$\boxed{\delta(t-t_0) \Leftrightarrow e^{-j\omega t_0}}$ delayed time impulse

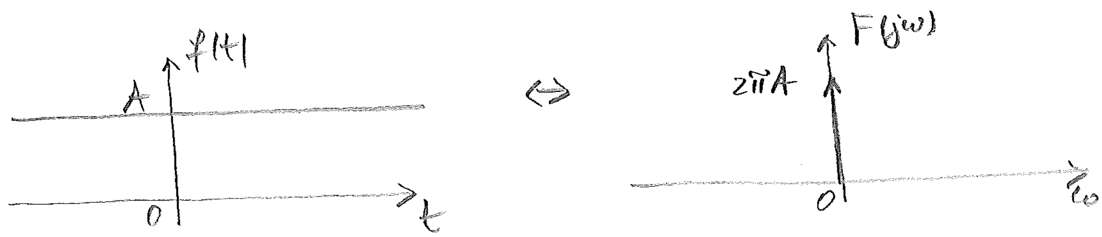
$\frac{1}{2\pi} e^{j\omega_0 t} \Leftrightarrow \delta(\omega-\omega_0)$ delayed frequency impulse

$\boxed{e^{j\omega_0 t} \Leftrightarrow 2\pi\cdot\delta(\omega-\omega_0)}$ complex exponential

$\boxed{A \Leftrightarrow 2\pi A\,\delta(\omega)}$ a constant function.

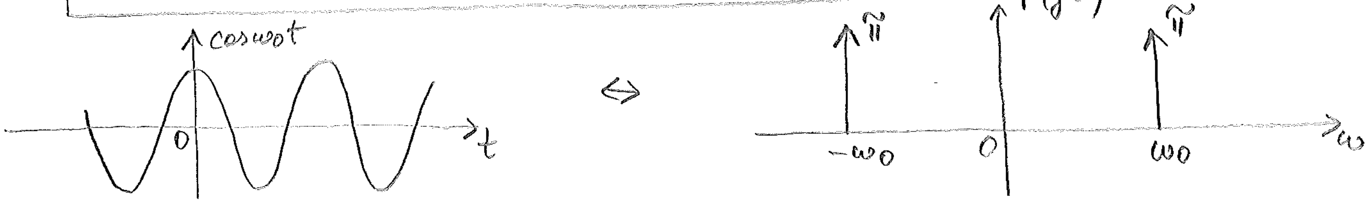


Fourier transform of an impulse is a constant.



Fourier transform of a constant is an impulse.

$\boxed{\cos\omega_0 t \Leftrightarrow \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]}$



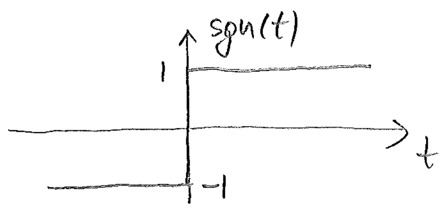
$\boxed{\sin\omega_0 t \Leftrightarrow j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]}$

Note Fourier transform introduced to deal with aperiodic signals can also be applied to periodic signals!

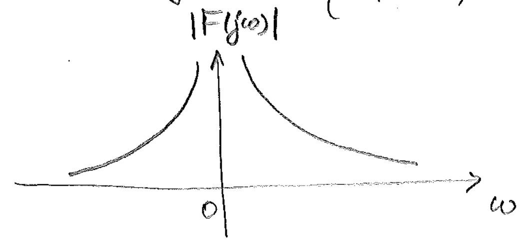
$$\boxed{e^{-at}\cdot u(t) \Leftrightarrow \frac{1}{a+j\omega}}$$ positive-time exponential

$$\boxed{e^{at}\cdot u(-t) \Leftrightarrow \frac{1}{a-j\omega}}$$ negative-time exponential

$$\boxed{\mathrm{sgn}(t) \Leftrightarrow \frac{2}{j\omega}}$$ where $\mathrm{sgn}(t) = \begin{cases} +1 & , t>0 \\ -1 & , t<0 \end{cases}$



$\Leftrightarrow$



$$\boxed{u(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

Relation to the Laplace transform

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{+\infty} f(t)\, e^{-st}\, dt$$

- To compare the two transforms, we must require that $f(t)$ be a <u>causal</u> or <u>positive-time</u> signal. Then letting $\boxed{s = \sigma + j\omega}$:

$$\mathcal{L}\{f(t)\cdot u(t)\} = \int_{0^-}^{+\infty} f(t)\cdot u(t)\cdot e^{-(\sigma+j\omega)t}\, dt = \int_{-\infty}^{+\infty} \left[f(t)\cdot u(t)\cdot e^{-\sigma t}\right]\cdot e^{-j\omega t}\, dt$$

$$\Rightarrow \boxed{\mathcal{L}\{f(t)\} = \mathcal{F}\{f(t)\cdot e^{-\sigma t}\}}$$

$$\boxed{\mathcal{F}\{f(t)\} = \mathcal{L}\{f(t)\}_{s=j\omega}}$$: holds only for functions for which the Fourier integral converges!

- Laplace transform of causal function $f(t)$ is the Fourier transform of the causal function $f(t) \cdot e^{-\sigma t}$!
- Or, Fourier transform of causal $f(t)$ is the Laplace transform of $f(t)$ calculated for $\sigma = 0$ (that is, on the $j\omega$ axis) !

Fourier transform applications

$V(j\omega) = Z(j\omega) \cdot I(j\omega)$

Fourier transforms of voltage and current.

$Z_R(j\omega) = R$

$Z_L(j\omega) = j\omega L$

$Z_C(j\omega) = \frac{1}{j\omega C}$

For a linear circuit, a forcing function $x(t)$, the response $y(t)$:

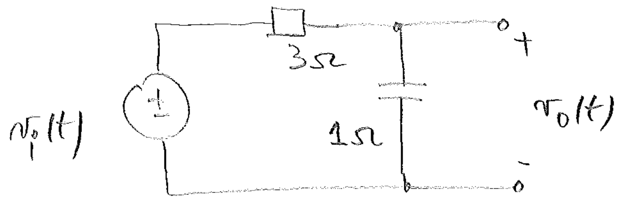
$$\boxed{Y(j\omega) = H(j\omega) \cdot X(j\omega)}$$

network/transfer function, can be found

- via ω-domain circuit analysis (now we work with Fourier transforms instead of phasors)
- via s-domain techniques, and then let $s \to j\omega$

- Fourier approach generalizes the phasor technique to aperiodic signals, subject to the constraint that they be Fourier transformable !

Example Use Fourier transform approach to find $v_0(t)$, when $v_i(t) = 10 \cdot \cos 2t$ [V]



NOTE: Solve, on your own, using the Phasor method.

$$V_0(j\omega) = \frac{\frac{1}{j\omega 1}}{3 + \frac{1}{j\omega 1}} \cdot V_i(j\omega) = \frac{1}{1 + j3\omega} V_i(j\omega) = H(j\omega) \cdot V_i(j\omega)$$

$$V_0(j\omega) = \frac{1}{1 + j3\omega} \cdot 10\pi \left[\delta(\omega+2) + \delta(\omega-2) \right]$$

Therefore:

$$\boxed{v_0(t)} = \mathcal{F}^{-1}\{V_0(j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 10\pi \frac{\delta(\omega-2) + \delta(\omega+2)}{1 + j3\omega} e^{j\omega t} dt =$$

$$= 5\left(\frac{e^{j2t}}{1+j6} + \frac{e^{-j2t}}{1-j6} \right) = \boxed{\frac{10}{\sqrt{37}} \cdot \cos(2t - 80.54°)} \text{ [V]}$$