Chapter 18 | Fourier analysis techniques

- We'll deal with **periodic** functions, which can be written as the **sum** of an infinite number of sine and cosine functions, which are harmonically related.

- Periodic functions: $$\boxed{f(t) = f(t \pm nT)} \quad (1), \quad n = 1, 2, 3, \dots$$
  T = period in seconds.

- Examples: $\sin \omega t$, $\cos \omega t$, $e^{j\omega t}$
  triangle, sawtooth, rectangle, pulse-train, etc.

→ implications:

(1) provides indication of how power associated with a periodic signal distributes among its harmonic components. (diagrams known as "power spectra")

(2) allows us to find the response to <u>any</u> periodic signal, regardless of its waveform. (response to individual sine components + superposition)

Angular frequency: $$\boxed{\omega_0 = \frac{2\pi}{T}} \quad (2)$$

1 Fourier series

: a periodic function $f(t)$ having period T can be expressed as an <u>infinite</u> summation of basic sinusoidal components.

$$\boxed{f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)} \quad (3)$$

: "amplitude-phase" Fourier series representation of $f(t)$.

Also, called the "cosine" Fourier series.

A_0: full-cycle average of f or DC value

A_n: amplitude; $n\omega_0$: angular frequency; θ_n: phase — of the nth sinusoidal component

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot \cos n\omega_0 t + b_n \cdot \sin n\omega_0 t \right)} \quad (4)$$

: "cosine-sine" Fourier series.

ω_0 = fundamental frequency

$n\omega_0$ = nth harmonic frequency

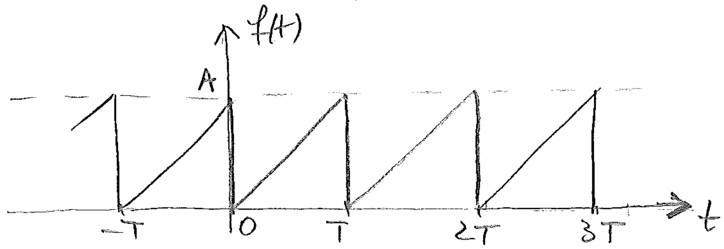
a_n, b_n = Fourier coefficients.

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, dt} \quad (5)$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cdot \cos n\omega_0 t \; dt} \quad (6)$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cdot \sin n\omega_0 t \; dt} \quad (7)$$

Example: Find the sine-cosine and the amplitude-phase Fourier series for sawtooth wave:



Select $t_0 = 0 \Rightarrow f(t) = \frac{A}{T} \cdot t$ for $0 < t \leq T$

By eq(5) $\Rightarrow a_0 = \frac{1}{T}\int_0^T \frac{A}{T} \cdot t\, dt = \frac{A}{T^2} \cdot \frac{t^2}{2}\Big|_0^T = \frac{A}{2}$

Also by eq. (6),(7):

$$a_n = \frac{2}{T}\int_0^T \frac{A}{T} \cdot t \cdot \cos n\omega_0 t \; dt = \frac{2A}{T^2}\left[\frac{\cos n\omega_0 t}{(n\omega_0)^2} + \frac{t \cdot \sin n\omega_0 t}{n\omega_0} \right]_0^T$$

$$= \frac{2A}{T}\left(\frac{\cos 2n\pi - 1}{(n\omega_0)^2} + \frac{T \cdot \sin 2n\pi - 0}{n\omega_0} \right) = 0$$

$$b_n = \frac{2}{T}\int_0^T \frac{A}{T}\cdot t\cdot \sin n\omega_0 t\, dt = \frac{2A}{T^2}\left[\frac{\sin n\omega_0 t}{(n\omega_0)^2} - \frac{t\cos n\omega_0 t}{n\omega_0}\Big|_0^T\right] =$$

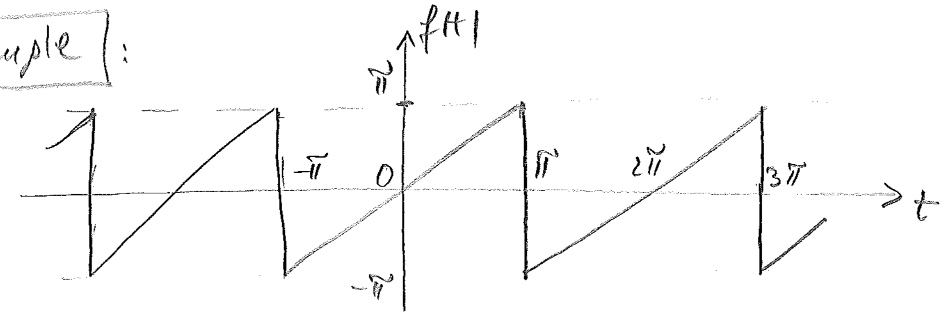
$$= \frac{2A}{T^2}\left(\frac{\sin 2n\pi - 0}{(n\omega_0)^2} - \frac{T\cos 2n\pi - 0}{n\omega_0}\right) = \frac{2A}{T^2}\left(-\frac{T}{n\omega_0}\right) = -\frac{A}{n\pi}$$

$$\Rightarrow \boxed{f(t) = A\left(\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin n\omega_0 t\right)}$$

<u>Also</u>: $f(t) = A\left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos(n\omega_0 t + 90^\circ)\right)$

$$\hookrightarrow f(t) = \frac{A}{2} - \frac{A}{\pi}\sin\omega_0 t - \frac{A}{2\pi}\sin 2\omega_0 t - \frac{A}{3\pi}\sin 3\omega_0 t - \cdots$$

Example:



$f(t) = t, \quad -\pi < t < \pi$

$f(t+2\pi) = f(t) \qquad T = 2\pi \Rightarrow \omega_0 = \frac{2\pi}{T} = 1$

Choose or select $\boxed{t_0 = -\pi}$.

By eq. (5) we get: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = 0$

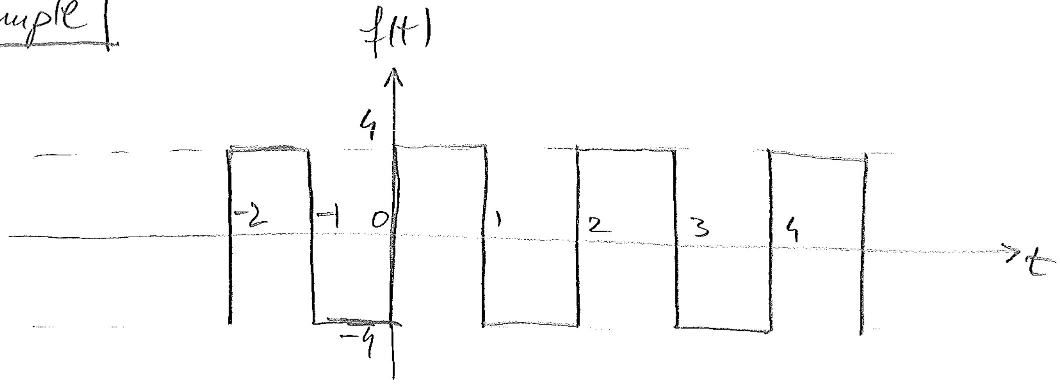
Also by eq. (6), (7) we get:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cdot\cos nt\,dt = \frac{1}{n^2\pi}\left(\cos nt + nt\cdot\sin nt\right)\Big|_{-\pi}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cdot\sin nt\,dt = \frac{1}{n^2\pi}\left(\sin nt - nt\cdot\cos nt\right)\Big|_{-\pi}^{\pi} = -\frac{2\cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}$$

$$\boxed{f(t) = 2\left(\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \cdots\right)}$$

Example



$$f(t) = \begin{cases} 4 & 0<t<1 \\ -4 & 1<t<2 \end{cases}$$

$f(t+2) = f(t)$; $T = 2 \Rightarrow \omega_0 = \pi$

NOTE: This is an $\boxed{\text{odd}}$ function, for which it can be shown that:

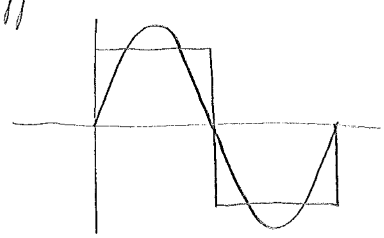
$$\boxed{a_n = 0} \qquad n = 0, 1, 2, \ldots$$

$$\boxed{b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t \, dt} \qquad n = 1, 2, \ldots$$

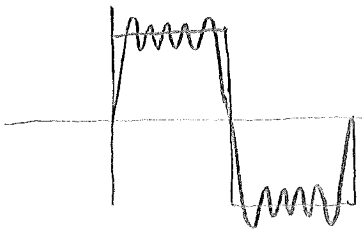
Therefore:

$$\begin{cases} a_n = 0 & n = 0, 1, 2, \ldots \\ b_n = \frac{4}{2}\int_0^1 4\cdot\sin n\pi t \, dt = \frac{8}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 & , n \text{ even} \\ \frac{16}{n\pi} & , n \text{ odd} \end{cases} \end{cases}$$

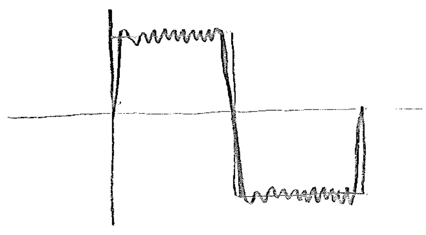
Approximations to the square wave:



: the fundamental

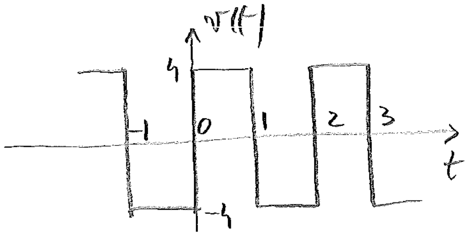


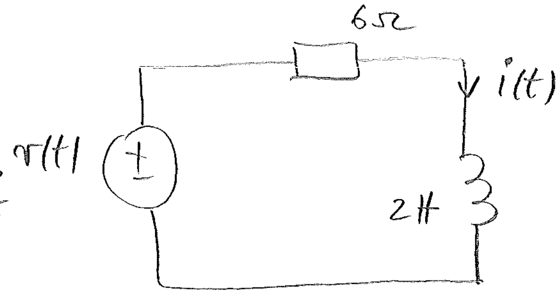
: the first <u>nine</u> harmonics.



: the first <u>19</u> harmonics.

Example **Response to periodic excitations**





$$v(t) = \begin{cases} 4 & 0 < t < 1 \\ -4 & 1 < t < 2 \end{cases}$$

$$v(t+2) = v(t)$$

- From previous example: $\boxed{v(t) = \dfrac{16}{\pi} \sum_{n=1}^{\infty} \dfrac{\sin(2n-1)\pi t}{2n-1}}$ [V]

$$\boxed{v(t) = \sum_{n=1}^{\infty} v_n}$$

where: $v_n = \dfrac{16}{(2n-1)\pi} \cdot \sin(2n-1)\pi t$

$$v_n = \frac{16}{(2n-1)\pi} \cdot \cos\left[(2n-1)\pi t - 90^\circ\right]$$

- Phasor based analysis: phasor voltage:

$$V_n = \frac{16}{(2n-1)\pi} \angle -90^\circ$$

- Impedance seen at the terminals of the source at ω_n is:

$$Z(j\omega_n) = 6 + j\,2(2n-1)\pi$$

$$= 2\sqrt{9 + \pi^2 (2n-1)^2} \cdot \angle \tan^{-1} \frac{(2n-1)\pi}{3}$$

- Hence, the phasor current:

$$\boxed{I_n = |I_n| \cdot \angle -90^\circ - \theta_n}$$

where: $\begin{cases} |I_n| = \dfrac{8}{(2n-1)\cdot \pi \sqrt{9 + \pi^2 (2n-1)^2}} \\ \theta_n = \tan^{-1} \dfrac{(2n-1)\pi}{3} \end{cases}$

– And the time-domain current is:

$$i_n(t) = |I_n| \cdot \cos(\omega_n t - 90° - \theta_n) = |I_n| \cdot \sin(\omega_n t - \theta_n)$$

– Finally, by superposition of all v_n (which make up the signal $v(t)$) we have:

$$\boxed{i(t) = \sum_{n=1}^{\infty} i_n(t) = \sum_{n=1}^{\infty} |I_n| \cdot \sin(\omega_n t - \theta_n)} \quad \text{(see figure\#1)}$$

$$= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\sqrt{9+\pi^2(2n-1)^2}} \cdot \sin\left[(2n-1)\pi t - \tan^{-1}\frac{(2n-1)\pi}{3}\right]$$

NOTES

– Current amplitude decreases as n increases. In the limit, it tends to zero as $n \to \infty$.

– So, the series may be truncated to a relatively few terms and still yield a good approximation!

– Work is minimized if we work using the "cosine" Fourier series form.

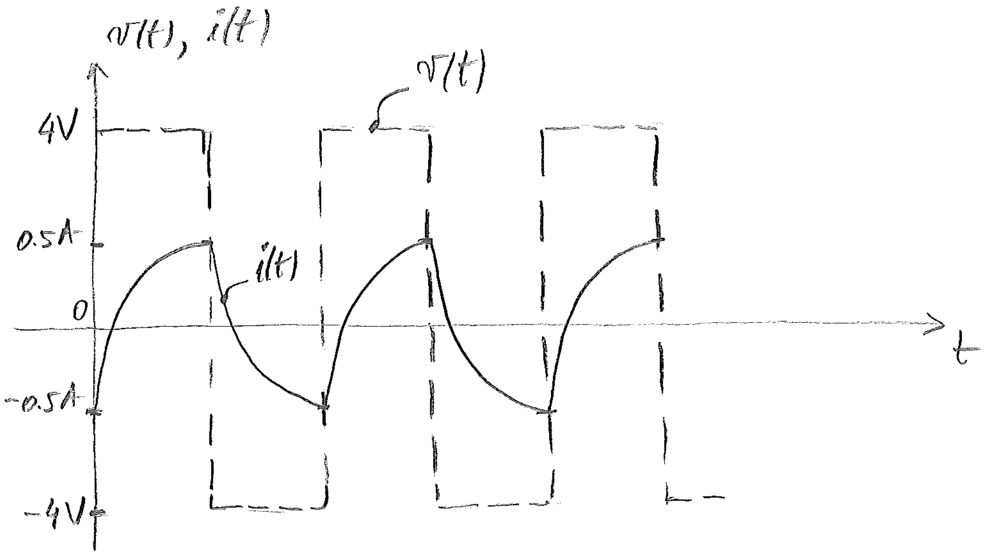


figure #1

The exponential Fourier series

- Using:

$$\begin{cases} \cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) \\ \sin n\omega_0 t = \frac{1}{j2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) \end{cases}$$

it can be shown that:

$$\boxed{f(t) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{jn\omega_0 t}} \quad (8)$$

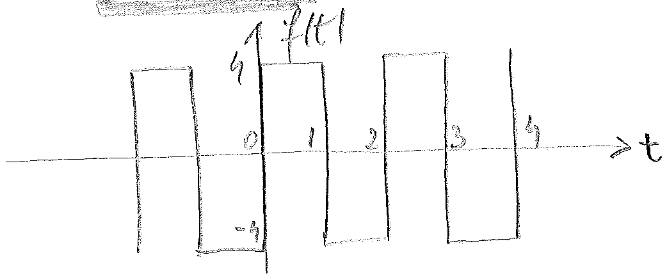
: the "exponential" Fourier series

NOTE: very useful, particularly in considering frequency response, one of the most important applications of Fourier series!

where:

$$\boxed{c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cdot e^{-jn\omega_0 t}\,dt \triangleq \frac{a_n - jb_n}{2}} \quad (9)$$

Example:



$$f(t) = \begin{cases} 4 & 0<t<1 \\ -4 & 1<t<2 \end{cases}$$

$f(t+2) = f(t) \Rightarrow T = 2 \Rightarrow \omega_0 = \frac{2\pi}{T} = \pi \quad \overset{(9)}{\Longrightarrow} \quad c_n = \frac{1}{2}\int_{-1}^{1} f(t)\, e^{-jn\pi t}\,dt$

- case 1: $\boxed{n \neq 0} \Rightarrow c_n = \frac{1}{2}\int_{-1}^{0}(-4)e^{-jn\pi t}\,dt + \frac{1}{2}\int_{0}^{1} 4e^{-jn\pi t}\,dt = \frac{4}{jn\pi}\left[1-(-1)^n\right]$

- case 2: $\boxed{n = 0} \Rightarrow c_0 = \frac{1}{2}\int_{-1}^{1} f(t)\,dt = 0$

- Therefore, the exponential Fourier series of $f(t)$ is: $\boxed{f(t) = \frac{8}{j\pi}\sum_{n=-\infty}^{+\infty} \frac{1}{2n-1}\cdot e^{j(2n-1)\pi t}}$