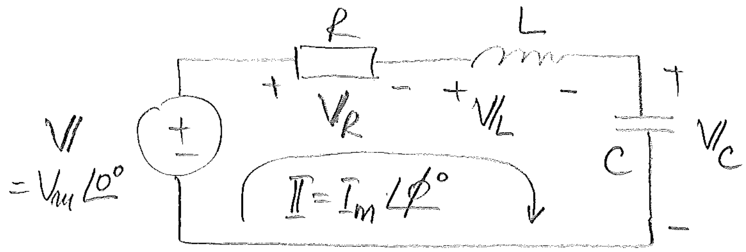
Chapter 16 | Frequency Response

1 | Series resonance | - inductance and capacitance connected in series.



- Impedance seen by the source is:

(1) $$\boxed{Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + jX}$$

(2) $\boxed{X = \omega L - \frac{1}{\omega C}}$ is the $\boxed{\text{net}}$ reactance of the circuit.

- Depending on the value of ω of the applied source, we have $\boxed{3}$ possibilities:

$\boxed{\text{Case 1}}$ $\omega L < \frac{1}{\omega C}$; $X < 0$ => Capacitive behavior.

$\boxed{\text{Case 2}}$ $\omega L > \frac{1}{\omega C}$; $X > 0$ => Inductive behavior.

$\boxed{\text{Case 3}}$ $\omega L = \frac{1}{\omega C}$; $X = 0$ => borderline. circuit behaves $\boxed{\text{resistively}}$ and I and V are in phase with each other.

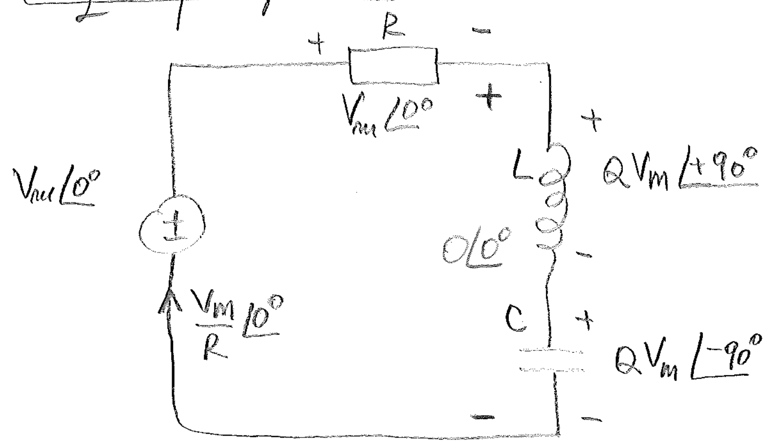
This behaviour is referred as "unity power-factor resonance" and occurs at the special frequency ω_0 that makes $\omega_0 L = \frac{1}{\omega_0 C}$ or:

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \quad \text{rad/sec}$$

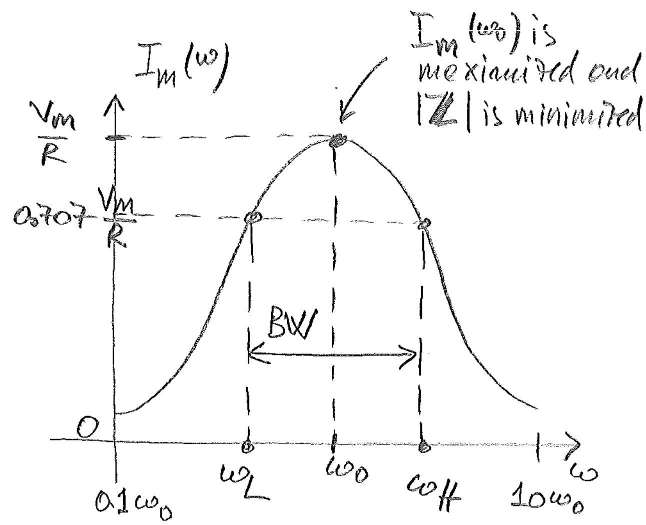
(3) called "resonance frequency"

Note: LC series combination act as <u>short-circuit</u> for ω_0.

Frequency response:



Series RLC at resonance



Band-pass function
(Bell-shaped profile)

- Using the notation:

$$\boxed{Q = \frac{1}{R\sqrt{C/L}}} \quad (4)$$

and called "quality factor"
reactances can be expressed as:

$$\omega L = RQ\frac{\omega}{\omega_0}$$

$$\frac{1}{\omega C} = RQ\frac{\omega_0}{\omega}$$

which can be used in equation (1) to get:

$$\boxed{Z = R\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]} \quad (5)$$

$$\mathbb{I} = \frac{\mathbb{V}}{Z} = \frac{\mathbb{V}}{R\left[1+jQ\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]} \quad (6)$$

from where:

$$\boxed{|\mathbb{I}| = I_m(\omega) = \frac{V_m}{R}\cdot\frac{1}{\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}} \quad (7)$$

$$\boxed{\measuredangle\mathbb{I} = \phi(\omega) = -\tan^{-1}\left[Q\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]} \quad (8)$$

- At resonance the <u>average power</u> P dissipated in the resistance R is also maximized:

$$P_{max} = \frac{1}{2}\frac{V_m^2}{R} = \frac{V_{rms}^2}{R} \qquad (9)$$

- Frequencies at which P is down to half of its maximum value are called "<u>half-power frequencies</u>" ω_L and ω_H.

- "<u>Half-power bandwidth</u>" is: $\boxed{BW = \omega_H - \omega_L}$ (10)

ω_L, ω_H can be found by solving equation:

$$1 + Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) = 2 \rightarrow \frac{\omega_L}{\omega_0} = \sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q}$$

$$\frac{\omega_H}{\omega_0} = \sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q}$$

$\Rightarrow$

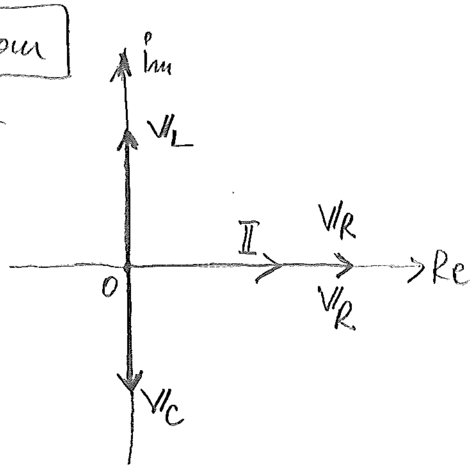
$$\Rightarrow \boxed{BW = \frac{\omega_0}{Q}} \qquad (11)$$

- Also: (3)+(4)+(11) $\boxed{BW = \frac{1}{L/R} = \frac{R}{L}}$ (12)

NOTE: In a series RLC circuit ω_0 is set by L and C and BW is set by R and L!

Phasor diagram

for additional insight!



$V_R = R I_m \angle \phi$

$V_L = \omega L I_m \angle \phi + 90°$

$V_C = \frac{1}{\omega C} \cdot I_m \angle \phi - 90°$

At resonance $\phi = 0°$.

- At resonance: $\mathbb{I} = \frac{V_m}{R} \angle 0^\circ$

and with $\omega = \omega_0$ we have:
$$\begin{cases} V_R = V_m \angle 0^\circ \\ V_L = Q \cdot V_m \angle 90^\circ \\ V_C = Q \cdot V_m \angle -90^\circ \end{cases}$$

- So, if $Q > 1 \Rightarrow$ peak voltages across the reactive elements will be greater than that of applied source!
- This phenomenon is called "<u>resonance voltage rise</u>"
- In the limit $R \to 0$, $Q \to \infty$ and we would have infinite voltage rise!

Energy at resonance

- instantaneous energies stored in the inductance and capacitance:

$$w_L(t) = \frac{1}{2} L i_L^2(t) \qquad w_C(t) = \frac{1}{2} C v_C^2(t)$$

- Total energy stored:

$w_L(t) + w_C(t)$ is maximum at resonance and called "<u>maximum stored energy</u>"

- At resonance:
$$\begin{cases} i_L(t) = \frac{V_m}{R} \cos \omega_0 t \\ w_L(t) = \frac{Q}{\omega_0} \cdot \frac{V_{rms}^2}{R} \cos^2 \omega_0 t \end{cases}$$

$$\begin{cases} v_C(t) = Q V_m \cdot \cos(\omega_0 t - 90^\circ) = Q V_m \sin \omega_0 t \\ w_C(t) = \frac{Q}{\omega_0} \cdot \frac{V_{rms}^2}{R} \cdot \sin^2 \omega_0 t \end{cases}$$

Hence: maximum stored energy is:

$$\boxed{w_L(t) + w_C(t) = \frac{Q}{\omega_0} \cdot \frac{V_{rms}^2}{R}} \quad (13)$$

which is time invariant!

- At resonance there is no energy exchang between source and the LC pair.
- L and C exchange their energy internally.
- The only energy that the source needs to supply is that dissipated by R.

- Of interest is "energy dissipated per cycle" at resonance : $\boxed{w_{R(cycle)}}$

$$\boxed{w_{R(cycle)} = \frac{P_{max}}{f_0}} \qquad \text{where } f_0 = \frac{\omega_0}{2\pi}$$

Devide energy dissipated in one second by the number of cycles contained in one second.

$$\boxed{w_{R(cycle)} = \frac{2\pi}{\omega_0} \cdot \frac{V_{rms}^2}{R}} \quad (14)$$

$$\frac{w_L(t) + w_C(t)}{w_{R(cycle)}} = \frac{Q}{2\pi}$$

$$\Rightarrow \boxed{Q = 2\pi \cdot \frac{\text{maximum stored energy}}{\text{energy dissipated per cycle}}} \quad (15)$$